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FOREIGN TECHNOLOGY DIVISION

INERTIAL GUIDANCE OF BALLISTIC MISSILES

by

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EDITED MACHINE TRANSLATION

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П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

The monograph discusses the mathematical principles of certain allowed variants of inertial guidance of the flight of ballistic missiles, i.e., the control without the use of any external information (radio signals, radiation of stars and others).

It is assumed that the providing of the assigned flight range of the missile is produced as a result of the well-timed switching off of its engine by a signal entering from the computer. Fed to the input of the latter are current readings of sensitive elements of the system of inertial guidance, which measure the apparent acceleration of the missile or integrals from the apparent time acceleration. The control should be such that the deviation of the actual motion of the missile from the rated does not have an effect on the range of its flight. For this measuring instruments of the system of inertial navigation are placed aboard the rocket in a quite definite manner, and, specifically, the direction of their axes of sensitivity in a number of cases is stabilized by means of gyroscopes. The setting of the necessary directions of axes of sensitivity and the search of relations which should satisfy current readings of sensing elements for the formation in the computer of the command for the switching off of the engine, and this is the main content of the book. Questions of the inertial guidance of motion of the missile in a lateral direction are also considered.

The monograph is intended for specialists in the field of the theory of control processes. It can prove to be useful in the investigation of new problems of this discipline and also as a training manual.

THE AUTHOR'S COMMENT

This book came about as a result of thoughts of many years over the general problems of inertial guidance systems of moving objects. Some of them are presented in my other monograph, published in 1963 under the name of "Mechanics of Gyroscopic Systems," in which, in accordance with its name, considerable space has been given also to the theory of sensing elements of systems of inertial guidance – a gyroscope in a gimbal suspension, gyroscopic stabilizer and gyroscopic integrator of the apparent unit accelerations. In this book, on the contrary, studied basically is the possibility of the solution of problems of inertial guidance of ballistic missiles by means of different combinations of gyroscopic and other mechanized sensing elements and computers. Of course, questions examined here exhaust by no means the whole diversity of ideas and possibilities of autonomous control of ballistic missiles. In recent years in the theory of specific inertial systems great successes have been achieved. They have not received as yet the proper reflection in monographic literature. It can be hoped that for the assimilation and development of new concepts of inertial control the proposed book proves to be useful.

The author thanks A. S. Kachanov, D. M. Klimov, D. F. Klim, I. S. Kovner and M. Ye. Temchenko for their valuable advice in the editing and preparation of the manuscript for press.

INTRODUCTION

Given in this small monograph are mathematical bases of one of the possible aspects of the theory of flight control of ballistic missiles without any external information (radio signals, radiation of the stars and others). The initial data for such control are readings of instruments located aboard the missile. They record both the orientation of the missile with respect to fixed directions (at fixed stars) and the difference in its motion from free flight without resistance of the medium. The effect of such instruments is based on the phenomenon of inertia. Because of this the mentione guidance is called inertial. Besides instruments which incorporate laws of mechanics, in systems of inertial guidance the application of instruments founded upon other physical principles (spins of elementary particles, standing waves of coherent radiation of lasers and others) is possible.

Creation of the variant examined in the monograph of the theory of inertial guidance is based on the assumption that aboard the missile there is stabilization (specifically, gyroscopic) of directions of axes of sensitivity of meters of apparent accelerations — newtonmeters or instruments directly measuring the time integral from current values of apparent acceleration. The latter are callec below integrators of accelerations; they can also be named impulsometers [Translator's note: This term is not verified]. The selection of these sensing elements was determined by the many years of interest of the author on gyroscopic instruments and other mechanica measuring devices.

The instruments have a different kind of error. The effect of the latter on the accuracy of flight of the missile requires a special investigation, which goes beyond the limits of this monograph. Here it is assumed that instruments of systems of inertial guidance are ideal, i.e., they function as though without errors. The same is referred to servodrives and reading devices and also any, which accomplish, specifically, arithmetical operations and the integration of current values of different parameters.

Airborne instruments and devices should not be excessively complex and bulky. Therefore, keeping in mind the use of simplified schemes, with the solution of problems of inertial guidance certain fundamental errors can be allowed. The latter, however, should not involve great deviations in the missile from its target. The selection of an appropriate compromise (at which deviations are small, and the system of inertial control is not too complex) constitutes one of the basic tasks of the designer of a specific system of inertial guidance.

Independently of the form of the guidance system, in its design one should proceed from the fact that the same target – the end of the free-flight section – can be hit by the ballistic missile, without necessarily moving along the programmed trajectory but according to an infinite set of other adjacent trajectories. Therefore, for an accurate hitting of the target it is not necessary that at the instant of termination of the active flight section of the missile its coordinates and component velocities in a certain system of coordinates, fixed relative to the earth, would be equal to the calculated provided by the program of control. The last observation is very important since the providing of the just mentioned equalities places before the system of control of flight of the missile very difficult, not always feasible tasks. The attaining, a rather accurate determination of current values of the coordinates themselves of the moving missile and projections of its velocity is considerably simpler. It is possible, therefore, to interrupt the powered-flight section of the missile at exactly that instant when the totality of deviations in its moving coordinates and projections

of velocity from the appropriate calculation values provides subsequent motion along one of the trajectories leading to the target. The determination of relationships which should be satisfied by the mentioned deviations at instant of the switching off of the engine, constitutes the main task of the stated theory of inertial guidance of flight of ballistic missiles. In this case without apparent deterioration of the accuracy of hitting of the missile on the target, considerable simplifications of guidance systems can be produced because of the separate control of the range and lateral motion of the missile and also due to the selection of specific directions of axes of sensitivity of inertial instruments and the forming of signals from them.

With gyroscopic stabilization of axes of sensitivity of newtonmeters and integrators of accelerations, it is natural to construct calculated equations and equations of inertial guidance in the system of coordinates not taking part in the rotation of the earth. Let us note, however, that range of flight of the missile is determined in this case not only by values of its coordinates and components of velocities relative to such a system at the instant of termination of the power-flight section, but also of duration of the latter.

In the computer of the given system of inertial guidance of flight range according to current readings of newtonmeters and integrators of accelerations, there is generated a certain alternating magnitude, uniquely connected at each instant with the magnitude of range of the missile. Understood by the latter is that magnitude of range which is obtained if the thrust of the engine at the instant of time suddenly becomes zero. To provide accurate hitting of the missile at the target, the switching off of the engine should be produced at the instant of time of achievement by the mentioned magnitude of assigned value. Since this magnitude is determined by the actual course of the change in coordinates and projections of the velocity of the missile, then it is called functional. As it will be shown below, it is possible to propose different forms of functionals. With their construction in the computer, subsequently,

there will not be taken into account the squares and products of deviations in the actual coordinates and projections of the velocity of the missile at the instant of end of operation of the engine from the rated. The same refers to deviations of moving coordinates and projections of the velocity of the rocket and also to the time interval between the necessary and calculated instants of the switching off of the engine. The assignment of the control of range is thus solved in a linear approximation. This should not lead to great errors in flight at the assigned range of the missile with well-controlled thrust of the engine and with the possibility of the switching off of the latter after the feed of the appropriate command with a minimum delayed pulse. In the case when control of the thrust is hampered, subsequent refinements of equations determining the termination of the power-flight section are necessary. These problems and also the problem of inertial control without the switching off of the engine are not considered in this monograph.

Results of investigations, given in this monograph indicate that in the flight of a missile over great ranges it is expedient to have on the gyroscopic stabilizer two integrators of acceleration. Axes of sensitivity of these integrators should be definitely oriented in two different directions. Fed into the computer, which determines the instant of the switching off of the engine, are the current value of one integrator and the result of complementary integration of readings of the other. In principle, it is possible to bypass one integrator of acceleration if in the process of flight of the missile properly both parameters of the integrator and direction of its axis of sensitivity are changed.

The use of two stabilized integrators of accelerations or one but with alternating parameters allows constructing the so-called complete functional most accurately resolving the problem of the control of range of the missile. Subsequent development of the theory leads, specifically, to the basis of the arrangement of integrators of accelerations directly aboard the missile under the condition of introduction into the guidance system of the so-called standard integrator of accelerations with accurate execution of its commands.

The acceleration of force of the earth's gravity depends on the distance between the missile and the center of the earth and also on geographical coordinates of the missile. Consequently, forces of gravity acting on the missile in its actual and calculated motions are not equal. This fact can condition the error in the determining of the necessary instant of switching off of the engine even during accurate operation of the integrators of accelerations and of gyroscopes of the system of inertial control. For the complete elimination of this error the airborne computer should be supplemented by a certain comparatively complex integrating equipment. However, a considerable decrease in this error can be achieved also by simpler means — because of a small change in the orientation of the integrator of acceleration.

Deviation of the rocket from the target in a lateral direction can be eliminated with the help of introduction into the system of inertial guidance of the so-called lateral integrator, which controls the motion of the rocket in the direction of the normal to assigned programmed plane of its flight. The distinction of the actual magnitude of duration of the free-flight section from its calculated value leads to additional lateral deviations of the missile because of the rotation of the earth. However, in principle it is possible to avoid errors of such a kind if we use current values of integrators of acceleration of system of range control.

Content of book is the following. Chapter I gives a solution to the problem of inertial guidance of ballistic missiles in a simplified formulation for the purpose of explaining basic ideas and methods examined in detail in subsequent chapters. Here the earth is taken as being flat and not rotating. The effect of the atmosphere is not considered and the force of gravity is considered constant in magnitude and direction. It is natural that under such simplifying assumptions the motion of the missile in the free-flight section of its flight is completely described by known equations of theoretical mechanics about the motion of a material particle in a vacuum in a uniform field of the force of gravity. Further it is explained

which relationship should be satisfied by small changes in parameters of the end of the power-flight section, i.e., coordinates of the missile and projections of its velocity at the instant of the switching off of the engine in order that it hits the assigned target with an error of not more than the second order of smallness. This relationship serves as the basic point for the formation of a certain variable, called the ballistic function. In the examined case the value of this function at any fixed instant of time is the approximate expression for the error in the range of the rocket, which will arise if the engine of the missile is turned off at precisely this instant. It is natural that for the hitting of the missile on the target the engine should be turned off at the instant of the passage of current values of this function through zero.

To construct a ballistic function aboard the rocket in the form of a certain electrical or mechanical magnitude with the variant of the inertial system selected in the monograph the presence of special sensing elements is proposed – newtonmeter and a computer, which contains in its composition integrating, multiplying and summing elements. In the example of the simplest meter of apparent acceleration is established the connection between the magnitude being measured, the real acceleration of the missile and acceleration of the force of gravity. Concepts of apparent velocity and apparent travel of the missile are introduced and it is indicated, specifically, that for the direct measurement of the apparent velocity it is necessary that the axis of sensitivity of the integrator of accelerations be stabilized. It turns out that the ballistic function, which is referred to the given case of motion of the missile in a uniform field of gravity without allowing for resistance on the side of the atmosphere, can be constructed aboard the missile, using only the integrator of acceleration with subsequent integration of its current readings by means of the computer. At the end of the chapter an analysis of this question is given both from an analytical and geometrical point of view. It is indicated that the flight range of the missile will be changed by a magnitude of the second order of smallness if the vector of its actual velocity at the end of the

power-flight section appears the same as that in the calculated case, and the position of the missile will be somewhat displaced in a certain definite direction, or, on the contrary, with a fixed position of the missile the difference between the real and calculated vectors of its velocity at the same instant of time proves to be definitely directed. In the examined case the mentioned directions coincide with each other. They, specifically, are parallel to the vector of velocity of the missile at instant of its hitting of the target with calculated motion. In general when the curvilinearity of the form of the earth, the effect of the atmosphere on the descending branch of the free-flight section of the trajectory and heterogeneity of the field of the earth's gravity are taken into account, such directions also exist, but they are not parallel. The perpendiculars to them, the so-called λ- and μ-directions can serve, as it is indicated in more detail in Chapter III, for orientation of axes of sensitivity of integrators of accelerations of the appropriate system of autonomous control of the flight range of the missile.

Chapter II gives ballistic functions of various kind for the general case of flight of the missile, i.e., without additional simplifying assumptions, which took place in Chapter I. Here it is impossible to express the magnitude of range of the missile in the form of a definite equation, which would contain parameters of the end of the power-flight section, for example in the starting system of coordinates. Nevertheless, correct to smallness of the second order relative to differences between current parameters of the real motion of the missile and parameters of the end of the power-flight section of its calculated motion, it is possible (with the help of the use of the same instruments as in Chapter I) to determine the error in the range which would take place with the switching off of the engine at the current instant of time. The expression for such an arbitrary error in range can be accepted as the ballistic function. Derivatives of the magnitude of flight range of the missile entering into this function, according to parameters of the end of the power-flight section, i.e., according to coordinates and projections of velocity, are taken at calculated values of the latter.

Thus they are constant quantities, which can be determined earlier for each assigned case of flight of the missile.

With the inertial control of range the current value of the ballistic function should be determined onboard the missile itself. This leads to the necessity of continuous determination by means of the computers of current coordinates of the missile and projections of its velocity according to readings of newtonmeters or, in other cases, integrators of apparent accelerations. The starting system for this goal proves to be barely adequate because of the necessity of the calculation of translational and coriolis accelerations. Therefore, it is more expedient to pass from coordinates and projections of velocity of the missile in the starting system of coordinates to appropriate magnitudes referring to the nonrotational system with the origin at the center of the earth. As a result, after the rejection of terms of the second order of smallness and terms dependent on the motion of the missile in a direction perpendicular to the plane of flight, the so-called initial ballistic function is formed, and they are linear with respect to the moving coordinates and projections of velocity of the missile and time. The expression for the initial ballistic function in a nonrotating system of coordinates contains a derivative of the magnitude of flight range of the rocket with respect to duration of the power-flight section. However, this derivative can be excluded from the analysis by means of the use of some auxiliary relation. The latter follows from the equality of the actual range to its calculated value, if coordinates and projections of velocity of the end of the power-flight section accurately coincide with current values of coordinates and projections of velocity of any particles of the calculated free-flight section. Finally the basic ballistic equation is obtained for determination of the necessary current time of the switching off of the engine in flight of the missile at the assigned range. Determination of current coordinates and projections of velocity of the missiles, which enter the left part of this equation, requires continuous integration aboard the missile of appropriate nonlinear differential equations, which connect the second derivatives of coordinates with

appropriate projections of the apparent acceleration and acceleration of the force of gravity. However, we can substantially simplify such a problem, if into the basic ballistic equation we substitute expressions of coordinates and projection velocities obtained as a result of the integration of approximate linear differential equations in the so-called isochronal variations of coordinates of the missile. The latter are differences between the acutal and calculated values of appropriate coordinates of the missile, which refer to the same instant of time. With the proper arbitrary extension of the calculated power-flight section of flight of the missile during the calculated instant of the switching off of the engine isochronal variations of coordinates, projections of velocity and projections of apparent acceleration can be considered small magnitudes during the whole interval of time which corresponds to the real power-flight section.

Differential equations for isochronal variations of coordinates are derived in Chapter III, and their approximate integration, allowing for terms appearing as a result of the heterogeneity of the field of the earth's gravity, is developed in Chapter IV. In Chapter III in differential equations for isochronal variations of coordinates, the mentioned terms are completely droped, whereupon variations of projections of velocity become equal to variations of projections of the apparent velocity of the missile, and variations of the coordinates themselves – to variations of projections of the apparent path. This allows converting the basic ballistic equation to one of the forms allowing the construction of current values of its left side directly onboard the missile according to current readings of integrators of accelerations.

Different orientation of axes of sensitivity of the integrators of accelerations are possible. Depending on this, the computing part of the system of inertial control of the range and the number of elements entering it are changed. For example, with orientation of axes of sensitivity according to invariable λ- and μ-directions, which was already mentioned above, it is possible to manage with only

one element of repeated integration of accelerations unlike the case of using two such elements necessary in the arrangement of axes of sensitivity in parallel to the axes of the nonrotating system of coordinates.

It proves to be possible to construct a ballistic equation even with the help of a single meter of acceleration, if we according to the earlier assigned law change the orientation of its axis. In the computer there should be provided in such a case the integration of current readings of this meter, preliminarily multiplied by the coefficient, which also is changed with the course of time.

The problem indicated above on the creation of a ballistic equation can be solved approximately also with the help of one integrator of acceleration, as this is indicated at the end of the chapter in the example of construction of the system of inertial control of range with accurate control of the direction of thrust of the engine according to indications of standard integrator of accelerations. In this case the axes of sensitivity of the standard integrator and integrator of the system of control of the range are located perpendicular to each other. Finally, if we consider the deviation in the force of thrust of the engine from the longitudinal axis of the missile to be insignificant, then for solution of the problem of control of range one can use current readings of the integrator, the housing of which is directly fastened to the side of the missile, and the axis of sensitivity is parallel to its longitudinal axis (so-called longitudinal integrator).

In Chapter IV there is formed the integration of the totality of differential equations for isochronal variations of coordinates of the missile with the approximate calculation in them of terms which characterize changes in projections of the force of gravity during motion of the missile according to the law different from the calculated. Corresponding to this, the basic ballistic equation derived out in Chapter II is converted to the form similar to that examined in Chapter III. Because of a specific method of integration

of the mentioned totality of equations, the problem on inertial control of the range is solved by the same means as it was earlier, i.e., with the help of current readings of integrators of accelerations and computers, which contain only integrating, multiplying and summing elements. This method is based on the preliminary replacement of equations of the mentioned system by equivalent integral differential and cumulative relations with the subsequent introduction into the terms being integrated instead of the sought functions, which represent current values of coordinates of the missiles approximating their polynomials of the second or third degree. Coefficients of approximating polynomials of the second degree are selected so that the polynomials would satisfy the same initial conditions as the sought functions, and, furthermore, coincide with the latter for the instant of termination of the power-flight section. With a more accurate solution to the problem, one should approximate the isochronal variations of coordinates by means of polynomials of the third degree. In this case time derivatives are equated also to appropriate projections of velocity of the missile at the instant of termination of the power-flight section. As a result for the determination of variations of coordinates and projections of velocity of the missile, algebraic equations with coefficients dependent on time are obtained. The latter, with sufficient accuracy, can be replaced by constants equal to values of these variations at the instant of termination of the power-flight section of flight of the missile. Finally the desired variations are expressed in terms of readings of integrators of accelerations and integrals of their time readings. Thereby, the subsequent creation of different forms of ballistic functions similar to those given in Chapter III, fundamental has no basic difficulties and is reduced basically to certain changes in coefficients and directions of axes of sensitivity of integrators of accelerations.

Finally, Chapter V of the book is devoted to the general problems of inertial control of the lateral motion of ballistic missiles. The removal of lateral deviation in the rocket from the target requires different methods than the providing of the assigned range of its flight, where the required accuracy can be obtained

because of the well-timed switching off of the engine. The duration of the power-flight section with flight at the same range depends for rockets of the same design on a number of random facts and, consequently, is itself a random magnitude, the mean value of which is near to the magnitude of duration of the power-flight section of the calculated motion. Therefore it is sufficient that the control of lateral motion of the missile would be especially accurate only during the interval of time in which switching off of the engine can occur.

The current removal of the missile from the programmed plane is connected with projections on the normal to this plane of its apparent acceleration and acceleration of the force of gravity by means of a differential equation of the second order. To control the lateral motion of the missile, it is necessary to know current values of this removal and its time derivative. They can be obtained by means of the so-called lateral integrator of accelerations and simplest computer. As a first approximation can in the mentioned differential equation drop terms containing the projection of the force of gravity, whereupon it is immediately integrated. The lateral departure of the rocket proves to be equal to the projection of its apparent path on the normal to the programmed plane of flight, and the value of velocity of this departure – respectively, to the projection of the apparent velocity. With the obtaining of more accurate expressions for the mentioned sought magnitudes, the approximation method of integration of differential equations given in Chapter IV is used.

In Chapter V it is also indicated how one can use readings of the instruments of inertial control of the range in order to remove the additional lateral deviation in the rocket from the target because of noncoincidence of its actual and calculated motions in the projection on the programmed plane.

In the appendix of the book an analytical derivation of cosines of angles between axes of system of coordinates with a finite turn is given. The table of cosines is used in Chapter II.

CHAPTER I

THEORY OF INERTIAL CONTROL OF FLIGHT RANGE OF THE BALLISTIC MISSILE IN SIMPLIFIED FORMULATION

§ 1. Equation Expressing Flight Range of the Missile in Terms of the Coordinate and Projection of Its Velocity at the End of the Power-Flight Section

The general formulation of the problem of control of flight range of the ballistic missile is most convenient to explain in the following simplest example.

Let us assume that the earth is flat and does not rotate, the atmosphere surrounding the earth is absent, and the acceleration of the force of the earth's gravity f, coinciding in this case with gravity g, is constant in magnitude and direction. Equations of the motion of the missile[1] on the free-flight section of its trajectory under such simplifying assumptions are completely integrated, and thereby the whole solution to the problem can be conducted in general form up to the end.

Let the power-flight section of flight of the missile proceed from the origin O of the fixed system of coordinates xy, the x axis of which is horizontal, and the y axis is directed along the vertical upward (Fig. 1). The initial velocity of the missile is considered equal to zero.

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[1]Here and further the term "motion of the missile" is understood as motion of its center of mass.

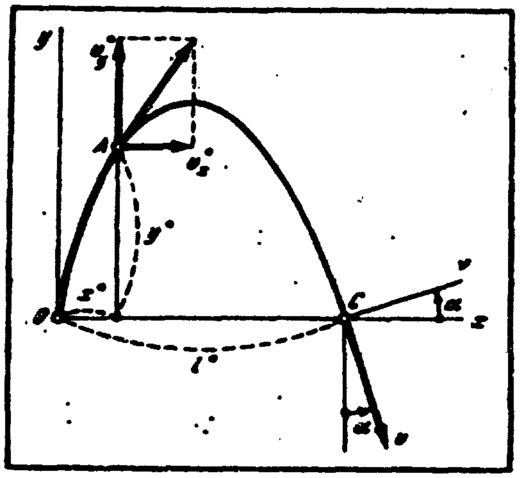

Fig. 1.

The *calculated* or *programmed* motion of the missiles is that one at which its coordinates are changed in accurate conformity with their precalculated values.

Let us designate coordinates of the real motion of the missile at instant σ, termination of the power-flight section of its flight, i.e., at the instant of complete switching off of the engine, by x and y, and the same coordinates in calculated motion – by x^* and y^* (Fig. 1). Let us designate by v_x and v_y projections on the axis x and y of the velocity of the rocket at the instant of the termination of the power-flight section of its real motion and respectively by v_x^* and v_y^* – their calculated values. The magnitude of duration of the power-flight section of the actual flight of the missile σ in general is distinguished from its calculated value σ^*.

The integration of equations of motion of the missile in the power-flight section of motion

$$\frac{d^2x(t)}{dt^2}=0, \qquad \frac{d^2y(t)}{dt^2}=-g \tag{1.1.1}$$

immediately leads to the following expressions for current projections of velocity $v_x(t)$ and $v_y(t)$:

$$v_x(t)=\frac{dx(t)}{dt}=v_x,\qquad v_y(t)=\frac{dy(t)}{dt}=v_y-gt, \tag{1.1.2}$$

and its moving coordinates $x(t)$ and $y(t)$:

$$x(t)=x+v_xt,\qquad y(t)=y+v_yt-\frac{gt^2}{2}. \tag{1.1.3}$$

Time t is read off here anew, i.e., from the instant of termination of the power-flight section and, consequently, from the beginning of the power-flight section of motion of the missile. Arbitrary constants with integration are selected so that there would take place the apparent equalities

$$v_x(0)=v_x,\quad v_y(0)=v_y,\quad x(0)=x,\quad y(0)=y. \tag{1.1.4}$$

Considering that the free-flight section is finished at instant $t = \tau$ at point C — intersection of the trajectory of the missile with the x axis, it is easy to determine the connection between the range of its flight l and appropriate parameters x, y, v_x and v_y — end of the power-flight section. Setting, first of all, in the second equation (1.1.3) $t = \tau$ and $y(\tau) = 0$, we obtain the following quadratic equation for determination of the duration of the free-flight section τ:

$$g\tau^2-2v_y\tau-2y=0. \tag{1.1.5}$$

Hence

$$\tau=\frac{1}{g}\left(v_y+\sqrt{v_y^2+2gy}\right), \tag{1.1.6}$$

since one takes the positive root of equation (1.1.5). Having substituted now the found value τ into the first equation (1.1.3), we obtain the desired expression for the range of the rocket as functions of parameters x, y, v_x and v_y — end of the power-flight section, namely:

$$l=x(\tau)=x+\frac{v_x}{g}\left(v_y+\sqrt{v_y^2+2gy}\right). \tag{1.1.7}$$

§ 2. Condition of Invariability of Flight Range with Little Distinction in Coordinates and Projections of Velocity of the Missile from Their Calculated Values at the Instant of Termination of the Power-Flight Section

Let us assume that the actual values of parameters of the power-flight section of flight of the missile x, y, v_x and v_y are distinguished from their calculated values by small magnitude

$$\Delta x = x - \overset{*}{x},\ \Delta y = y - \overset{*}{y},\ \Delta v_x = v_x - \overset{*}{v}_x,\ \Delta v_y = v_y - \overset{*}{v}_y. \tag{1.2.1}$$

The flight range of the rocket l, in accordance with that given in § 1, and, specifically, according to equation (1.1.7), is a function of magnitudes x, y, v_x and v_y, i.e.,

$$l = l(x,\ y,\ v_x,\ v_y). \tag{1.2.2}$$

In accordance with equality (1.2.1), we have

$$l = l(\overset{*}{x} + \Delta x,\ \ \overset{*}{y} + \Delta y,\ \ \overset{*}{v}_x + \Delta v_x,\ \ \overset{*}{v}_y + \Delta v_y). \tag{1.2.3}$$

Expanding now the right side of equation (1.2.3) into Taylor series for the function of many variables, we obtain correct to smallness of the first order relative to magnitude Δx, Δy, Δv_x and Δv_y

$$l = l(\overset{*}{x}, \overset{*}{y}, \overset{*}{v}_x, \overset{*}{v}_y) + \frac{\partial l}{\partial x}\Delta x + \frac{\partial l}{\partial y}\Delta y + \frac{\partial l}{\partial v_x}\Delta v_x + \frac{\partial l}{\partial v_y}\Delta v_y. \tag{1.2.4}$$

Partial derivatives of function $l(x,\ y,\ v_x,\ v_y)$ with respect to variables x, y, v_x and v_y are themselves functions of the same variables. In this case they can be presented in evident form as a result of the differentiation of function (1.1.7) according to the appropriate variable, namely:

$$\frac{\partial l}{\partial x} = 1,\ \frac{\partial l}{\partial y} = \frac{v_x}{\sqrt{v_y^2 + 2gy}},\ \frac{\partial l}{\partial v_x} = \frac{1}{g}\left(v_y + \sqrt{v_y^2 + 2gy}\right),$$

$$\frac{\partial l}{\partial v_y} = \frac{v_x}{g}\left(1 + \frac{v_y}{\sqrt{v_y^2 + 2gy}}\right). \tag{1.2.5}$$

According to rules of the composition of Taylor series, in equation (1.2.4) arguments of aforementioned partial derivatives should be assumed correspondingly equal to magnitudes x^*, y^*, v_x^* and v_y^*. Thus they are for the selected calculated motion of the missile certain constants. Taking into account, furthermore, that expression

$$l^* = l(x^*, y^*, v_x^*, v_y^*) \tag{1.2.6}$$

is the calculation flight range of the missile, let us transform equation (1.2.4) to the form

$$\Delta l = \Delta x + \frac{v_x^*}{\sqrt{(v_y^*)^2 + 2gy^*}} \Delta y + \frac{1}{g}\left(v_y^* + \sqrt{(v_y^*)^2 + 2gy^*}\right) \Delta v_x + $$
$$+ \frac{v_x^*}{g}\left(1 + \frac{v_y^*}{\sqrt{(v_y^*)^2 + 2gy^*}}\right) \Delta v_y, \tag{1.2.7}$$

where the difference

$$\Delta l = l - l^* \tag{1.2.8}$$

is the change in flight range of the missile because of noncoincidence of magnitudes of parameters of the end of the power-flight section x, y, v_x and v_y with their calculated values x^*, y^*, v_x^* and v_y^*.

The expression for Δl, allowing for equation (1.1.6), can be represented now in the form

$$\Delta l = \Delta x + k\Delta y + \tau^* (\Delta v_x + k \Delta v_y), \tag{1.2.9}$$

where

$$k = \frac{v_x^*}{\sqrt{(v_y^*)^2 + 2gy^*}}. \tag{1.2.10}$$

Let us note, that coefficient k has a simple geometric meaning. It is easy to show, using equations (1.1.2) and (1.1.6), that it is equal to the magnitude of the angular coefficient of the normal ν to the calculated trajectory of motion of the missile at point C, its intersection with the x axis (Fig. 1). Thus,

$$k = \operatorname{tg} \alpha, \qquad (1.2.11)$$

where α – angle between the mentioned direction ν and x axis (and also between the vertical line and vector of velocity of the missile at the instant of its incidence on the earth).

From equation (1.2.9) it follows that correct to smallness of the first order inclusively the range of the missile remains the very same if changes in parameters Δx, Δy, Δv_x and Δv_y – end of its power-flight section in comparison with their calculated values, will be subordinate to the condition

$$\Delta x + k\Delta y + \tau(\Delta v_x + k\Delta v_y) = 0. \qquad (1.2.12)$$

It is important that this condition is the linear relation with constant coefficients known earlier for the selected flight of the missile.

The duration of the power-flight section of flight σ, in contrast to parameters of its end x, y, v_x and v_y, does not play any role in the determination of the magnitude of range l. Therefore, there is no meaning to strive with execution of the specific program of motion of the missile for the rated value of duration σ^* of the power-flight section. On the contrary, by slightly lengthening or shortening the power-flight section, it is possible to obtain a fulfilling of just the given condition (1.2.12) and, consequently, provide the calculated range of the missile at not exactly an accurate sequence of the actual coordinates to their programmed values.

§ 3. Error in Flight Range of the Missile Presented in the Form of a Function of the Duration of the Power-Flight Section

Let us assume that, as also in § 1, $x(t)$ and $y(t)$ are functions which are the actual (not calculated) change in time of coordinates of the missile on the power-flight section, a $v_x(t)$ and $v_y(t)$, respectively, are projections of its velocity, and the instant $t = 0$ at this time corresponds to the beginning of the power-flight section.

Let us substitute into equation (1.2.9) for Δl quantities Δx, Δy, Δv_x and Δv_y respectively by differences

$$x(t)-x^*,\ y(t)-y^*,\ v_x(t)-v_x^*,\ v_y(t)-v_y^*. \tag{1.3.1}$$

As a result let us obtain the time function

$$\begin{aligned}\varepsilon(t)=x(t)-x^*+k[y(t)-y^*]+\tau^*[v_x(t)-v_x^*]+\\+k\tau^*[v_y(t)-v_y^*],\end{aligned} \tag{1.3.2}$$

which we call *ballistic* function.

At the instant of the switching off of the engine $t = \sigma^*$ this function, according to equation (1.2.9), turns into Δl and, consequently, determines the error in flight range of the rocket because of an accurate execution of the program of the power-flight section but accurate observance of its duration. However, if we interrupt the power-flight section at that instant when the ballistic function $\varepsilon(t)$ turns into zero, then the error into the range proves to be a magnitude of the second order of smallness, and the desired accuracy of flight of the missile will be achieved. Hence it follows that the problem of inertial control of the flight range of the missile can be reduced to the construction of current values of the ballistic function directly aboard the missile and to the switching off of its engine with the passage of function $\varepsilon(t)$ through zero. For this purpose, besides the summing and multiplying devices, the presence aboard the missile of special instruments which measure the apparent acceleration with subsequent integration of their current readings, or instruments directly recording the time integral from the apparent accelerations – integrators of accelerations is necessary. Meters of apparent accelerations will subsequently be called *newtonmeters*.

§ 4. Connection Between the Acceleration of the Missile in Its Motion Relative to the Fixed System of Coordinates and Reading of Newtonmeter Set on It

In the system of control of the flight range of the ballistic

missile when using newtonmeter one should keep in mind that the latter can measure only the apparent and not the real acceleration of that place of the missile where they are located. The apparent acceleration is usually called the difference between the acceleration of any points in the fixed system of coordinates and the acceleration of the force of gravity. Specifically, standard uniaxial newtonmeters should measure the projection a_ν of the apparent acceleration on its axis of sensitivity ν, i.e., magnitude

$$a_\nu = w_\nu - f_\nu \tag{1.4.1}$$

where w_ν – projection on axis ν of the real acceleration of the newtonmeter relative to the fixed system of coordinates and f_ν – projection on the same axis of acceleration of the force of the earth's gravity.

Let us explain equation (1.4.1) in the example of the newtonmeter, the sensing element of which is a small weight of mass m attached to the end of the spring of rigidity c (Fig. 2). The other end of the spring is sealed in the housing of the newtonmeter. The small weight can be moved within the housing on straight line ν, which is the axis of sensitivity of such a meter of apparent acceleration.

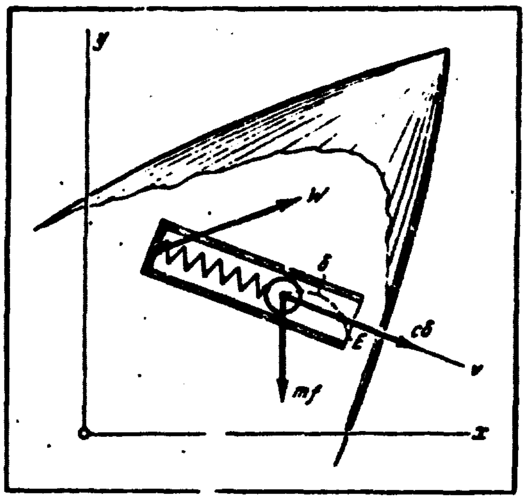

Fig. 2.

The equation of motion of the small weight relative to the housing of the newtonmeter, if we disregard the mass of the spring and friction of small weight about the internal cylindrical surface of the housing, can be presented in the form

$$-m\frac{d^2\delta}{dt^2}+mw_\nu^e=mf_\nu+c\delta. \tag{1.4.2}$$

Here δ – movement of the small weight from that position E at which the spring has not been stretched; w_ν^e – projection on direction ν of translational acceleration, i.e., acceleration with respect to the fixed system of coordinates of that place of the housing where at the given instant the small weight is located (projection of the coriolis acceleration to direction ν is equal to zero).

With translational movement of the housing of the newtonmeter

$$w_\nu^e=w_\nu, \tag{1.4.3}$$

where w_ν – acceleration of the place of fastening of the spring to the housing.

If, however, the housing of the newtonmeter has, furthermore, angular motions, then equality (1.4.3) should be considered as approximate. However, because of the comparatively small dimensions of newtonmeters the difference between w_ν^e and w_ν is important.[1]

Let us assume that the frequency p of natural oscillations of the small weight, determined by equation

$$p=\sqrt{\frac{c}{m}}, \tag{1.4.4}$$

[1]It is possible to show that this difference is equal to the product $\omega_0^2\rho$ where ω_0 – projection of angular velocity of the housing of the newtonmeter on the plane perpendicular to the axis ν and ρ – distance between the small weight and point of attachment of the spring.

is sufficiently great. Then the amplitude of these oscillations, conditioned basically by the change in acceleration of the housing of the newtonmeter w_ν will be small. In this case the term $-m\frac{d^2\delta}{dt^2}$ in equation (1.4.2) can be dropped, and allowing for equality (1.4.3) we can obtain relation

$$mw_\nu = mf_\nu + c\delta. \tag{1.4.5}$$

The member $c\delta$ of equation (1.4.5) is the elastic force of the spring. Deformation δ (see Fig. 2) will be considered positive if the small weight is displaced in a negative direction of axis ν. When $\delta = 0$ the spring is not stretched, and the small weight is located at position E (see Fig. 2). The observable shift in the small weight from position E can be graduated so that it would directly measure a certain magnitude a_ν connected to the deformation of the spring by relation

$$a_\nu = \frac{c}{m}\delta. \tag{1.4.6}$$

Replacing here the magnitude of deformation of the spring δ by its expression, by the following from equation (1.4.5), we will arrive at equation (1.4.1). Thus, the shift in the small weight is proportional to the projection of its apparent acceleration on the direction ν, i.e., on the axis of sensitivity of the newtonmeter.

The current reading of the integrator of the apparent accelerations or simply integrator of accelerations (it can also be called *impulsometer*) is a time integral from the projection of the apparent acceleration on its axis of sensitivity, i.e., magnitude

$$V_\nu(t) = \int_0^t a_\nu(t)\,dt. \tag{1.4.7}$$

If the axis of sensitivity of the integrator of acceleration retains a fixed direction with respect to the fixed system of coordinates, then

$$w_\nu = \frac{dv_\nu(t)}{dt}, \qquad (1.4.8)$$

where $v_\nu(t)$ – projection on the direction ν of velocity of the housing of the integrator in the same fixed system. Therefore, substituting into equation (1.4.7) the expression for a_ν from equation (1.4.1) and using equality (1.4.8), we obtain the relation

$$V_\nu(t) = v_\nu(t) - v_\nu(0) - \int_0^t f_\nu(t)\,dt, \qquad (1.4.9)$$

which connects the projection of velocity $v_\nu(t)$ with the reading of the integrator of acceleration $V_\nu(t)$. The latter is called in this case the projection of the apparent velocity on direction ν.

Function $f_\nu(t)$ which stands under the sign of the integral in the right side of relation (1.4.9) when the earth is not proposed to be flat, is changed with the course of time because of a change in the position of the missile relative to the earth.

With alternating orientation of the axis of sensitivity ν, for example, when the integrator of acceleration is located directly aboard the missile, equations (1.4.8) and (1.4.9), of course, are not applicable, and the reading of the integrator of acceleration is no longer equivalent to the corresponding projection of the apparent velocity.

§ 5. Construction of a Ballistic Function by Means of Two Integrators of Accelerations and a Computer

In this section let us examine the use in the system of inertial control of the flight range of two integrators of accelerations. Let us assume that the integrators are stabilized so that the axis of sensitivity of one of them during the whole power-flight section of the missile would remain parallel to the horizontal x axis, and the axis of sensitivity of the other would be directed parallel to the vertical y axis.

In the examined case, i.e., under the assumption of a nonrotating flat earth, projections of accelerations of the force of gravity on the x and y axes are expressed by equations

$$f_x = 0, \qquad f_y = -g. \tag{1.5.1}$$

Consequently, in accordance with formula (1.4.1), projections of the apparent acceleration on the same axis are the quantities

$$a_x = w_x(t), \qquad a_y = w_y(t) + g, \tag{1.5.2}$$

where

$$w_x(t) = \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2}, \quad w_y(t) = \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2}, \tag{1.5.3}$$

are corresponding projections of the real acceleration of the missile in the fixed system of coordinates xy. Here $x(t)$ and $y(t)$, just as earlier, are current coordinates of the missile on the power-flight section of its flight.

The current readings of the integrators with axes of sensitivity parallel to axes of coordinates x and y, according to equation (1.4.7), are quantities

$$V_x = \int_0^t a_x(t)\,dt, \qquad V_y = \int_0^t a_y(t)\,dt, \tag{1.5.4}$$

which are projections of the *apparent velocity* of the missile, respectively, on the axis of the fixed orientation x and y. In virtue of equations (1.5.2) and (1.5.3) they are connected with projections $v_x(t)$ and $v_y(t)$ of the real velocity of the missile relative to the system of coordinates xy by relation

$$V_x(t) = v_x(t), \qquad V_y(t) = v_y(t) + gt. \tag{1.5.5}$$

Into the latter constants of integration are absent, since projections of the real velocity of the missile at the initial instant

of the power-flight section, i.e., when $t = 0$, are equal to zero. Coordinates of the missile $x(t)$ and $y(t)$ at this instant are also equal to zero. Therefore, as a result of the integration with respect to time of current indications $V_x(t)$ and $V_y(t)$ of corresponding integrators of accelerations will be reduced on the basis of equations (1.5.3) and (1.5.5) to the following relation:

$$S_x(t)=x(t),\qquad S_y(t)=y(t)+\frac{gt^2}{2}. \tag{1.5.6}$$

In them time functions

$$S_x(t)=\int_0^t V_x(t)\,dt,\qquad S_y(t)=\int_0^t V_y(t)\,dt \tag{1.5.7}$$

can be called projections of the *apparent path* of the missile.

By means of relations (1.5.5) and (1.5.6) one can express coordinates $x(t)$ and $y(t)$ in terms of functions $S_x(t)$ and $S_y(t)$ and projections $v_x(t)$ and $v_y(t)$ of its real velocity – in terms of current readings of integrators of accelerations $V_x(t)$ and $V_y(t)$. If further we substitute these results into equation (1.3.2) for the ballistic function $\varepsilon(t)$, then after the simplest conversions we will obtain for it the following expression:

$$\begin{aligned}\varepsilon(t)={}&S_x(t)+kS_y(t)+\tau^0[V_x(t)+kV_y(t)]-\\&-x^0-ky^0-\tau^0(v_x^0+kv_y^0)-\tau^0kgt-k\frac{gt^2}{2},\end{aligned} \tag{1.5.8}$$

where the coefficient k, as before, is determined by equation (1.2.10).

For the construction aboard the missile of a current value of the ballistic function $\varepsilon(t)$ in the form of an alternating mechanical or electrical magnitude in the examined case the presence of two integrators of accelerations, clocks and computer is necessary. The composition of the latter, specifically, should include two elements of additional integration of readings of integrators of accelerations

§ 6. Control of Flight Range of the Missile by Means of a Single Integrator of Accelerations with Inclined Orientation of Its Axis

An attentive examination of expression (1.5.8) for the ballistic function $\varepsilon(t)$ leads to the conclusion that sums

$$V_x(t)+kV_y(t), \qquad S_x(t)+kS_y(t), \tag{1.6.1}$$

entering into its composition can be formed aboard the missile by means of only the integrator of accelerations with subsequent additional time integration of its instant readings. The axis of sensitivity ν of such an integrator should be inclined during the whole time of the power-flight section of flight of the missile to the horizontal axis x at a constant angle α, the magnitude of which is connected with coefficient k by relation (1.2.11). Thus (see § 2 of this chapter), the direction of the axis of sensitivity of the mentioned integrator is perpendicular to the vector of velocity of the missile at the instant of reaching by it of the target C with calculated motion (Fig. 1).

Let us note, first of all (Fig. 3), that for projections on the direction ν of the acceleration of the force of gravity and real acceleration of the missile we have equations

$$f_\nu=-g\sin\alpha, \qquad w_\nu=w_x(t)\cos\alpha+w_y(t)\sin\alpha. \tag{1.6.2}$$

Consequently, according to equation (1.4.1), we have

$$a_\nu=w_\nu(t)-f_\nu(t)=w_x(t)\cos\alpha+[w_y(t)+g]\sin\alpha \tag{1.6.3}$$

or, taking into account equality (1.5.2),

$$a_\nu(t)=a_x(t)\cos\alpha+a_y(t)\sin\alpha. \tag{1.6.4}$$

The last relation indicates that the apparent acceleration is a vector whose components are parallel to axes of coordinates x and y, respectively equal to quantities $a_x(t)$ and $a_y(t)$.

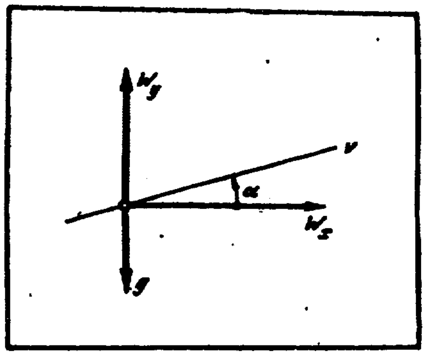

Fig. 3.

Integrating expression (1.6.4) for the projection of the apparent acceleration on the direction ν with respect to time, and taking into account equations (1.4.7) and (1.5.4), we obtain equality

$$V_\nu(t) = V_x(t)\cos\alpha + V_y(t)\sin\alpha, \tag{1.6.5}$$

which indicates that quantity $V_\nu(t)$ is in this case a projection on the direction ν of a certain vector with components $V_x(t)$ and $V_y(t)$ along axes x and y.

Function $V_\nu(t)$ is the current reading of the integrator of acceleration whose axis of sensitivity has the direction ν. Integrating this time function and taking into account equation (1.5.7), we obtain a new relation

$$S_\nu(t) = S_x(t)\cos\alpha + S_y(t)\sin\alpha, \tag{1.6.6}$$

in which quantity

$$S_\nu(t) = \int_0^t V_\nu(t)\,dt \tag{1.6.7}$$

can be called the projection of the vector of apparent path of the missile on direction ν.

Relations (1.6.5) and (1.6.6) can be given with the help of equation (1.2.11) the following form:

and

$$V_{\nu}(t) = \cos\alpha\,[V_x(t) + kV_y(t)] \tag{1.6.8}$$

$$S_{\nu}(t) = \cos\alpha\,[S_x(t) + kS_y(t)]. \tag{1.6.9}$$

From the obtained relations it follows that sums (1.6.1) and, consequently, and expression (1.5.8) for the ballistic function $\varepsilon(t)$ can be constructed aboard the missile by means of the use of current readings of the single integrator of acceleration and of its time integral. Really, on the basis of relations (1.6.8) and (1.6.9), expression (1.5.8) is converted to the form

$$\begin{aligned}\varepsilon(t) = \frac{1}{\cos\alpha}\,[S_{\nu}(t) + \tau^{\circ}V_{\nu}(t)] - (x^{\circ} + ky^{\circ}) - \\ - \tau^{\circ}(v_x^{\circ} + kv_y^{\circ}) - \tau^{\circ}kgt - k\frac{gt^2}{2}.\end{aligned} \tag{1.6.10}$$

As was already mentioned above, the duration of the power-flight section is determined by equation $\varepsilon(t) = 0$. In accordance with equations (1.6.10) and (1.2.11), this equation, which will subsequently be called *ballistic*, is reduced to the following, more convenient form:

$$\begin{aligned}V_{\nu}(t) + \frac{1}{\tau^{\circ}}S_{\nu}(t) = v_x^{\circ}\cos\alpha + (v_y^{\circ} + gt)\sin\alpha + \\ + \frac{1}{\tau^{\circ}}\left[x^{\circ}\cos\alpha + \left(y^{\circ} + \frac{gt^2}{2}\right)\sin\alpha\right].\end{aligned} \tag{1.6.11}$$

§ 7. Conversion of the Ballistic Equation by Using a New Rectangular System of Coordinates

The ballistic equation (1.6.11) acquires a more transparent form, if we introduce a new fixed system of coordinates $\xi\eta$ with the same origin as that for the system xy. Axis ξ of this system of coordinates is directed parallel to straight line ν, i.e., at the same angle α to the horizon (to the x axis) at which the axis of sensitivity of the integrator of accelerations should be located (Fig. 4). Then axis η will be the antiparallel to the vector of velocity of the missile at the instant of its hitting of the target with calculated motion (see Fig. 1).

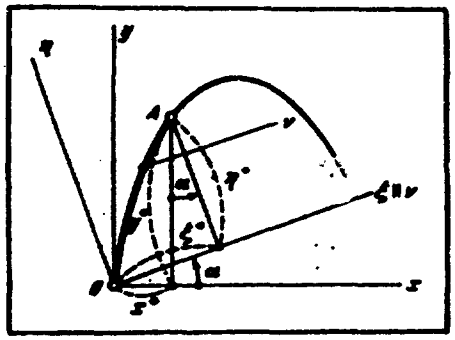

Fig. 4.

Coordinates of the end of the calculated power-flight section of the trajectory of the missile in the new system of coordinates are designated by ξ^* and η^*. They are connected with coordinates x^* and y^* of the same points in the xy system by equations of conversion

$$\xi^* = x^*\cos\alpha + y^*\sin\alpha, \qquad \eta^* = -x^*\sin\alpha + y^*\cos\alpha. \tag{1.7.1}$$

Let us designate projections of velocity of the missile on the axis ξ and η with its calculated motion at the instant of the switching off of the engine by v_ξ^* and v_η^*. They are connected with projections of the same velocity on the axes x and y, with quantities v_x^* and v_y^*, by similar equations

$$v_\xi^* = v_x^*\cos\alpha + v_y^*\sin\alpha, \qquad v_\eta^* = -v_x^*\sin\alpha + v_y^*\cos\alpha. \tag{1.7.2}$$

For moving coordinates of the missile in the system $\xi\eta$, which refers to its real motion, let us introduce designations $\xi(t)$ and $\eta(t)$, for projections of its velocity on these axes — $v_\xi(t)$ and $v_\eta(t)$, and, finally, for projections of acceleration — $w_\xi(t)$ and $w_\eta(t)$.

Since the axis ξ and direction v are parallel, then the projection of acceleration of the missile on the direction v is the equation

$$w_v(t) = w_\xi(t) = \frac{d^2\xi(t)}{dt^2}, \tag{1.7.3}$$

and, consequently, according to equations (1.4.1) and (1.6.2), we obtain

$$\frac{d^2\xi(t)}{dt^2} = a_\nu(t) + f_\xi = a_\nu(t) - g\sin\alpha. \qquad (1.7.4)$$

At the initial instant (t = 0) of the power-flight section, coordinates of the missile $\xi(t)$ and $\eta(t)$ and also projections of its velocity $v_\xi(t)$ and $v_\eta(t)$ are equal to zero. Therefore, successively integrating expression (1.7.4) for $d^2\xi(t)/dt^2$ with respect to time, we obtain equations

$$\begin{aligned} v_\xi(t) &= \frac{d\xi(t)}{dt} = V_\nu(t) - gt\sin\alpha, \\ \xi(t) &= S_\nu(t) - \frac{gt^2}{2}\sin\alpha, \end{aligned} \qquad (1.7.5)$$

where $V_\nu(t)$ and $S_\nu(t)$, as before, are projections of the apparent velocity of the missile and its apparent path on direction ν or, which is the same, on axis ξ. They are connected with the projection of the apparent acceleration $a_\nu(t)$ by relations (1.4.7) and (1.6.7).

When using equations (1.7.5), (1.7.2) and (1.7.1) the ballistic equation (1.6.11) for determination of the instant of the switching off of the engine is reduced to the following simple form:[1]

$$v_\xi(t) + \frac{1}{\tau}\xi(t) = v_\xi^* + \frac{1}{\tau}\xi^*. \qquad (1.7.6)$$

Not entering into equation (1.7.6) is either the coordinate of the missile $\eta(t)$ or projection $v_\eta(t)$ of its velocity on axis η. This means that the small distinction in the mentioned magnitudes at the end of the power-flight section from their calculated values η^* and

[1]Equation (1.7.6) can be obtained directly from the ballistic equation $\varepsilon(t)$ = 0, in which function $\varepsilon(t)$ is taken in the initial form (1.3.2). In order to be convinced of this, it is enough to substitute in expression (1.3.2) coefficient k by its value following from the relation (1.2.11) and to use equations (1.7.1) and (1.7.2) and also those similar to them for quantities $\xi(t)$ and $v_\xi(t)$.

v_{η}^{*} correct to smallness of the second order should not affect the flight range of the missile. In essence this is explained by the fact that for the control of the flight range of the missile it is enough to use only the integrator of accelerations whose axis of sensitivity is parallel to the axis ξ or, which is the same, as direction ν.

§ 8. Geometric Examination of the Condition of Hitting of the Missile on the Assigned Target

From a geometric point of view the ballistic equation (1.7.6) of the previous section can be given the following interpretation. Let us assume that the motion of the missile is such (Fig. 5) that at the instant of the switching off of the engine it proves to be at point A', shifted with respect to points A — the end of the calculated trajectory of the power-flight section on a small segment parallel of axis η. If its velocity at this instant appears the same as that at the end of the power-flight section of the calculated motion, then the trajectory of the missile in the free-flight section will have the same form as that in the calculated case. In order to obtain this trajectory, it is enough to shift by the magnitude of segment AA' the whole calculated trajectory forward in the direction of axis η or, which is the same, in the direction of the tangent to the trajectory at point C of its intersection with axis x (see Fig. 5). It is obvious that the point of intersection of the "shifted" trajectory with the x axis will be remote from points C, i.e., from the calculated points of fall of the rocket, on a smallness of the second order.

Let us assume that, on the contrary, at the end of the power-flight section the rocket arrives accurately at the calculated point A, but at a velocity somewhat distinguished from the calculated in magnitude and in direction (Fig. 6 and Fig. 7). The form of the trajectory proves to be different, and in general the rocket will no longer hit the assigned target.

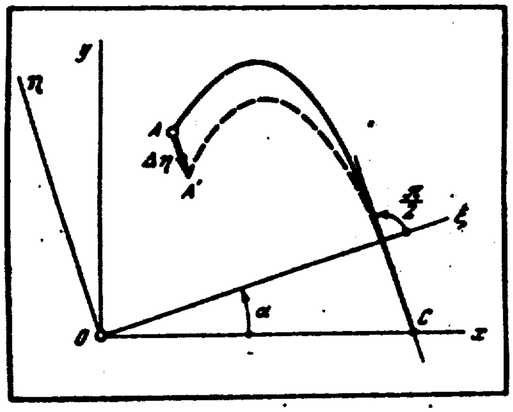

Fig. 5.

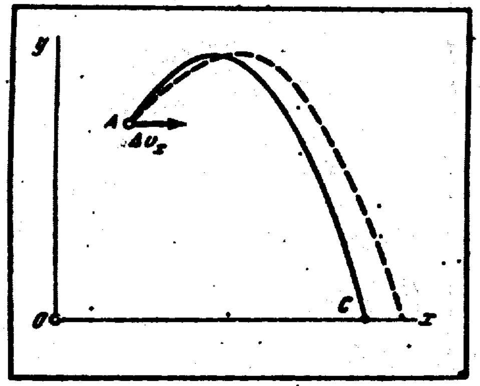

Fig. 6.

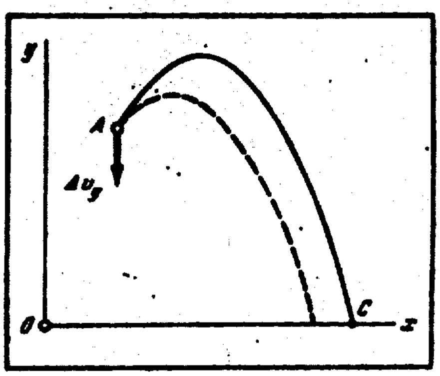

Fig. 7.

Nevertheless, as follows from the ballistic equation in the form (1.7.6), if the velocity of the missile will be different from the calculated by a small vector parallel of axis η (Fig. 8), then the deviation of the rocket from the target will also be a magnitude of the second order of smallness. Thus, in the case of a nonrotating flat earth deprived of an atmosphere, it is possible to show such coinciding directions of small vectors of the change in velocity or the shift of the rocket at the end of the power-flight section, in the presence of which the change in flight range of the rocket has the magnitude of the second order of smallness. Similar directions can be shown in general if one considers the rotation of the earth and its nonspherical state in the presence of an atmosphere. However, they prove to be different, and for the solution of the problem on

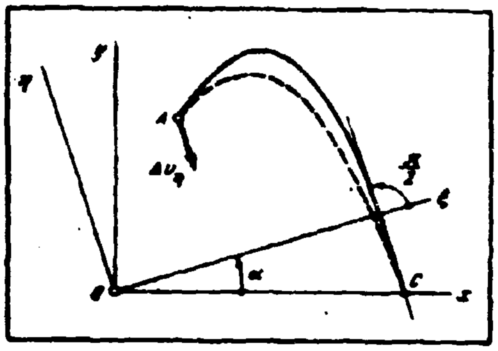

Fig. 8.

selection of the proper instant of the switching off of the engine of one integrator of accelerations with the axis of sensitivity of constant direction, in general, they are insufficient. The mentioned problem (also correct to smallness of the second order) is most simply solved with the help of two integrators of accelerations (see § 3 of Chapter III) with additional integration of readings of one of them.

CHAPTER II

THEORY OF INERTIAL CONTROL OF FLIGHT RANGE OF BALLISTIC MISSILES IN THE GENERAL FORMULATION

§ 1. Expression of the Error in Flight Range of the Missile with Small Changes in Parameters of the End of the Power-Flight Section. Ballistic Function

The theory of inertial control becomes incomparably more complex if with the motion of the missile, in contrast to the simplifying assumptions accepted in Chapter I, we take account of the change in acceleration of the force of gravity both in magnitude and in direction, consider the earth no longer to be fixed, and in the calculation of the free-flight section take account of the resistance of the atmosphere. The basic difficulty consists here in the selection of a certain rather simple function of parameters measured aboard the missile by inertial sensing elements. The function should be such that with the achievement by it of the earlier assigned value, it was possible to produce a switching off of the engine, having provided the calculated range of the missile. An example of a similar kind was given in Chapter I. This function, called subsequently, just as in Chapter I, ballistic (sometimes it is called *controlling* and also *controlling functional*), is constructed aboard the missile by means of the computer, which uses current readings of the integrators of accelerations. The magnitude of the ballistic function should be directly connected with the error in the flight range of the rocket which would occur if the switching off of the engine occurred at the current instant of time. The

system of inertial control in turn should give a signal for cessation of the operation of the engine upon achievement by the ballistic function of a value which corresponds to the turning of the mentioned error into zero.

One of the ballistic functions is the error itself in determining the flight range of the missile, expressed in terms of its moving coordinates $x(t)$, $y(t)$, $z(t)$ and projections $v_x(t)$, $v_y(t)$, $v_z(t)$ of velocity relative to the so-called *starting* system of coordinates xyz connected with the rotating earth. In this case it is proposed, of course, that at precisely the instant t complete switching off of the engine occurs.

It is convenient to put the beginning of starting system at the center of the earth, having directed the y axis along the radius of the earth through the points of start O and the x axis so that the coordinate plane xy, called the ***program***, contains the assigned points of fall of the missile C. If we direct the x axis to the side of points C, then direction of the z axis is thereby completely determined (Fig. 9). It is useful to note, that the trajectory of the missile does not lie in plane xy, passing, nevertheless, in the calculated case through the origin of coordinates O and through point C.

For the magnitude of the flight range of the missile l we can take the shortest geographical distance between the point of start O and actual point of fall of the missile D, and for its lateral deviation b – distance of points D to the programmed plane, i.e., plane xy (Fig. 9).

Let us introduce brief designations

$$\begin{aligned} x(\sigma) &= x, & y(\sigma) &= y, & z(\sigma) &= z, \\ v_x(\sigma) &= v_x, & v_y(\sigma) &= v_y, & v_z(\sigma) &= v_z \end{aligned} \tag{2.1.1}$$

for values of coordinates of the missile and projections of its velocity relative to the starting system of coordinates xyz at the instant of the switching off of the engine $t = \sigma$. In this case it

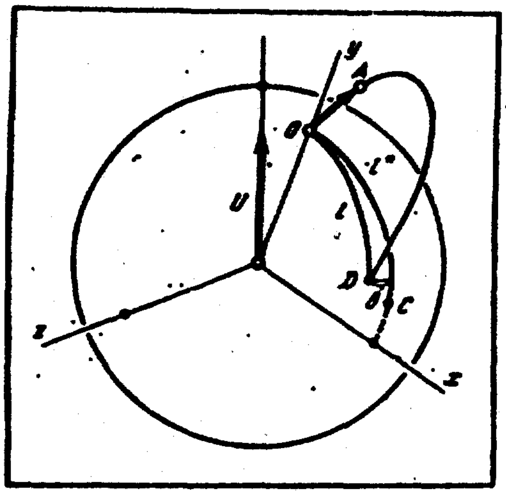

Fig. 9.

is considered that the instant $t = 0$ is referred to the beginning of motion of the missile from the start. With fixed values x, y, z and v_x, v_y, v_z range l depends on the width of the place of the start and on the location of the plane xy relative to the point of the compass. The position at the instant of termination of the active section of the celestial bodies – moon and sun relative to the earth, practically does not govern the flight range of the missile. A certain small effect on the magnitude of the range of the missile proves to be the state of the atmosphere, especially near the points of its fall, which subsequently, however, is not considered.

Having in mind any specific position of start of the missile and its assigned point of fall, we take that range l is only a function of coordinates x, y, z and projections of velocity v_x, v_y, v_z of the missile at the end of the power-flight section of flight and does not depend on the duration of the latter. Thus, we will consider that

$$l = l(x, y, z, v_x, v_y, v_z). \tag{2.1.2}$$

The construction of functions $l(x, y, z, v_x, v_y, v_z)$ in general in the form of an explicit function of their variables is impossible, since, in contrast to that given in Chapter I, the equation of motion

of the missile in the free-flight section in quadratures are not integrated. Nevertheless, mathematical machines with great accuracy allow comparatively rapidly calculating the magnitude of range of the missile l according to data of its coordinates x, y, z and projections of velocity v_x, v_y, v_z, which corresponds to the end of the power-flight section in the starting system of coordinates xyz.

Let us designate by x^*, y^*, z^* and v_x^*, v_y^*, v_z^* those values of coordinates and of projections of velocity of the missile in the starting system of coordinates xyz, which corresponds to the end of the power-flight section in the calculated motion and by l^* the calculated range of its flight. In accordance with equation (2.1.2), we have

$$l^* = l(x^*, y^*, z^*, v_x^*, v_y^*, v_z^*). \tag{2.1.3}$$

Let us form expression

$$\Delta l = l[x(t), y(t), z(t), v_x(t), v_y(t), v_z(t)] - l(x^*, y^*, z^*, v_x^*, v_y^*, v_z^*). \tag{2.1.4}$$

It can be accepted as the ballistic function mentioned in the beginning of this section. Actually, if we turn off the engine at any arbitrary instant t, then value of the selected function corresponding to this instant is directly the error in the flight range of the missile. If, however, the engine is turned off at that instant $t = \sigma$ at which this function turns into zero, then, naturally, there will not be an error in the flight range.

Because of the absence of the equation which expresses function (2.1.2) in an explicit form in terms of its arguments, the practical use of the difference (2.1.4) as a ballistic function is difficult. Considerably simpler is another ballistic function, which is obtained from the expansion of function l into Taylor series about calculated values of its arguments, i.e., about the totality of quantities x^*, y^*, z^*, v_x^*, v_y^*, and v_z^*. According to expression (2.1.4), retaining only smallness of the first order, we have

$$\Delta l = [x(t) - x^*]\frac{\partial l}{\partial x} + [y(t) - y^*]\frac{\partial l}{\partial y} + [z(t) - z^*]\frac{\partial l}{\partial z} + \\ + [v_x(t) - v_x^*]\frac{\partial l}{\partial v_x} + [v_y(t) - v_y^*]\frac{\partial l}{\partial v_y} + [v_z(t) - v_z^*]\frac{\partial l}{\partial v_z}. \qquad (2.1.5)$$

Here the partial derivatives $\frac{\partial l}{\partial x}, \frac{\partial l}{\partial y}, \frac{\partial l}{\partial z}, \frac{\partial l}{\partial v_x}, \frac{\partial l}{\partial v_y}, \frac{\partial l}{\partial v_z}$, called *ballistic coefficients*, are taken at values of their arguments, respectively equal to x^*, y^*, z^*, v_x^*, v_y^* and v_z^*, and, consequently, for the assigned flight of the rocket are constant quantities. The ballistic coefficients can be calculated, specifically, on high-speed computers.

It is obvious that with a small deviation in the motion of the missile from calculated as a ballistic function one can take expression

$$\delta(t) = [x(t) - x^*]\frac{\partial l}{\partial x} + [y(t) - y^*]\frac{\partial l}{\partial y} + [z(t) - z^*]\frac{\partial l}{\partial z} + \\ + [v_x(t) - v_x^*]\frac{\partial l}{\partial v_x} + [v_y(t) - v_y^*]\frac{\partial l}{\partial v_y} + [v_z(t) - v_z^*]\frac{\partial l}{\partial v_z}. \qquad (2.1.6)$$

Actually, with the switching off of the engine at the instant when this expression becomes equal to zero, the difference between the real and calculated flight range of the missile proves to be a magnitude of the second order of smallness.[1] In this case deviations in the parameters of the actual motion of the missile from the calculated at the end of the power-flight section of its flight are taken as smallness of the first order.

§ 2. Equations Connecting Coordinates of the Missile in Starting and Nonrotating System of Coordinates

To solve problems of inertial guidance, the starting system of coordinates represents certain inconveniences. Specifically, the determination of coordinates and components of velocities of the

[1]If we do not take into account, of course other facts disturbing the accuracy of flight of the missile, specifically, the effect of atmospheric conditions at the end of the power-flight section.

missile in this system according to readings of newtonmeters or integrators of accelerations is complicated by the necessity of calculation of translational and coriolis forces of inertia. Let us introduce, therefore, the nonrotating system of coordinates $\xi\eta\zeta$ with the same beginning at the center of the earth as that for the starting system xyz, which rotates together with the earth. Let us assume that at the instant of the beginning of motion of the missile $t = 0$ the axes of system of coordinates $\xi\eta\zeta$ and xyz respectively coincide.

Let us designate by U_ξ, U_η and U_ζ projections of angular velocity of the earth on corresponding axes of the system of coordinates $\xi\eta\zeta$ and by U_x, U_y and U_z – on axes of the system xyz. For an arbitrary instant of time the following equalities are valid

$$\begin{aligned} &U_\xi = U_x = lU, \qquad U_\eta = U_y = mU, \\ &U_\zeta = U_z = nU, \end{aligned} \tag{2.2.1}$$

where l, m, n – direction cosines of the earth's axis and, consequently, vector U in the system of coordinates $\xi\eta\zeta$ or, which is the same, in the system of coordinates xyz.

After the time t after the start of the missile, the system of coordinates xyz will be turned counterclockwise at angle

$$\varphi = Ut \tag{2.2.2}$$

relative to the system $\xi\eta\zeta$, if we observe rotation on the side of the positive direction of the vector of angular velocity of the earth U (i.e., on the side of the star Polaris).

Together with the current coordinates of the missile $x(t)$, $y(t)$ and $z(t)$ in the starting system of coordinates xyz, let us introduce its coordinates $\xi(t)$, $\eta(t)$, $\zeta(t)$ in the system $\xi\eta\zeta$. Axes of the latter, as follows from the aforementioned, do not change their orientation relative to directions at fixed stars.

The table of cosines of angles between the axes of system of coordinates $\xi\eta\zeta$ and xyz[1] is the following:

$$
\begin{array}{cccc}
 & \xi & \eta & \zeta \\
x & (1-\cos\varphi)l^2+\cos\varphi & (1-\cos\varphi)ml+n\sin\varphi & (1-\cos\varphi)nl-m\sin\varphi, \\
y & (1-\cos\varphi)lm-n\sin\varphi & (1-\cos\varphi)m^2+\cos\varphi & (1-\cos\varphi)nm+l\sin\varphi, \\
z & (1-\cos\varphi)ln+m\sin\varphi & (1-\cos\varphi)mn-l\sin\varphi & (1-\cos\varphi)n^2+\cos\varphi.
\end{array}
\tag{2.2.3}
$$

According to table (2.2.3), one can determine coordinates of the missile in the system $\xi\eta\zeta$, i.e., quantities $\xi(t)$, $\eta(t)$, and $\zeta(t)$, if direction cosines l, m, n, angle ϕ and coordinates $x(t)$, $y(t)$, $z(t)$ are known. The appropriate equations have the form

$$
\begin{aligned}
\xi(t) &= [(1-\cos\varphi)\,l^2+\cos\varphi]\,x(t)+[(1-\cos\varphi)lm- \\
&\quad -n\sin\varphi]\,y(t)+[(1-\cos\varphi)\,ln+m\sin\varphi]\,z(t), \\
\eta(t) &= [(1-\cos\varphi)\,ml+n\sin\varphi]\,x(t)+[(1-\cos\varphi)\,m^2+ \\
&\quad +\cos\varphi]\,y(t)+[(1-\cos\varphi)\,mn-l\sin\varphi]\,z(t), \\
\zeta(t) &= [(1-\cos\varphi)\,nl-m\sin\varphi]\,x(t)+[(1-\cos\varphi)\,mn+ \\
&\quad +l\sin\varphi]\,y(t)+[(1-\cos\varphi)\,n^2+\cos\varphi]\,z(t).
\end{aligned}
\tag{2.2.4}
$$

The reverse conversion, which expresses coordinates $x(t)$, $y(t)$ and $z(t)$ in terms of $\xi(t)$, $\eta(t)$ and $\zeta(t)$, corresponds to the finite turn of the system of coordinates $\xi\eta\zeta$ relative to the system xyz at the same angle ϕ but in the opposite direction. Specifically, it is obtained from the previous equations (2.2.4) if we replace in them angle ϕ by $-\phi$ and exchange places of quantities ξ and x, η and y, and also ζ and z. As a result we obtain equations

$$
\begin{aligned}
x(t) &= [(1-\cos\varphi)\,l^2+\cos\varphi]\,\xi(t)+[(1-\cos\varphi)\,ml+ \\
&\quad +n\sin\varphi]\,\eta(t)+[(1-\cos\varphi)\,nl-m\sin\varphi]\,\zeta(t), \\
y(t) &= [(1-\cos\varphi)\,lm-n\sin\varphi]\,\xi(t)+[(1-\cos\varphi)\,m^2+ \\
&\quad +\cos\varphi]\,\eta(t)+[(1-\cos\varphi)\,nm+l\sin\varphi]\,\zeta(t), \\
z(t) &= [(1-\cos\varphi)\,ln+m\sin\varphi]\,\xi(t)+[(1-\cos\varphi)mn- \\
&\quad -l\sin\varphi]\,\eta(t)+[(1-\cos\varphi)\,n^2+\cos\varphi]\,\zeta(t).
\end{aligned}
\tag{2.2.5}
$$

---

[1]See, specifically, the Appendix on page 98.

which, of course, can be obtained directly, using table (2.2.3).

§ 3. Projections of the Velocity of the Missile Relative to the Nonrotating System of Coordinates

Earlier there were introduced designations $v_x(t)$, $v_y(t)$ and $v_z(t)$ for current values of projections on axes x, y and z of velocity of the missile relative to the starting system of coordinates xyz rotating together with the earth. Projections of velocity of the missile in nonrotating system of coordinates $\xi\eta\zeta$ on the axis of this system are designated, respectively, by $u_\xi(t)$, $u_\eta(t)$ and $u_\zeta(t)$. It is obvious that

$$u_\xi(t)=\frac{d\xi(t)}{dt},\quad u_\eta(t)=\frac{d\eta(t)}{dt},\quad u_\zeta(t)=\frac{d\zeta(t)}{dt}, \tag{2.3.1}$$

in the same way as

$$v_x(t)=\frac{dx(t)}{dt},\quad v_y(t)=\frac{dy(t)}{dt},\quad v_z(t)=\frac{dz(t)}{dt}. \tag{2.3.2}$$

Quantities $v_x(t)$, $v_y(t)$ and $v_z(t)$ can be expressed by $u_\xi(t)$, $u_\eta(t)$, $u_\zeta(t)$ and inversely. According to equations of kinematics relative to the motion of the points, we have

$$\begin{aligned} u_x(t) &= v_x(t)+U_y z(t)-U_z y(t),\\ u_y(t) &= v_y(t)+U_z x(t)-U_x z(t),\\ u_z(t) &= v_z(t)+U_x y(t)-U_y x(t), \end{aligned} \tag{2.3.3}$$

where $u_x(t)$, $u_y(t)$, $u_z(t)$ – projections on the axes x, y and z of the velocity of the missile in the rotating system of coordinates $\xi\eta\zeta$.

In equations (2.3.3), in accordance with equalities (2.2.1), let us replace U_x, U_y and U_z, and coordinates $x(t)$, $y(t)$ and $z(t)$ will be expressed by means of relations (2.2.5) in terms of coordinates $\xi(t)$, $\eta(t)$ and $\zeta(t)$. If further projections $u_x(t)$, $u_y(t)$

and $u_z(t)$ are expressed, using table (2.2.3), in terms of projections $u_\xi(t)$, $u_\eta(t)$ and $u_\zeta(t)$, then we will arrive at the following desired equations

$$\begin{aligned}
v_x(t) = {} & [(1-\cos\varphi)\, l^2 + \cos\varphi]\, u_\xi(t) + \\
& + [(1-\cos\varphi)\, lm + n\sin\varphi]\, u_\eta(t) + \\
& + [(1-\cos\varphi)\, ln - m\sin\varphi]\, u_\zeta(t) - U_\eta\, \{[(1-\cos\varphi) nl + \\
& + m\sin\varphi]\, \xi(t) + [(1-\cos\varphi)\, mn - l\sin\varphi]\, \eta(t) + \\
& + [(1-\cos\varphi)\, n^2 + \cos\varphi]\, \zeta(t)\} + U_\zeta\, \{[(1-\cos\varphi)\, lm - \\
& - n\sin\varphi]\, \xi(t) + [(1-\cos\varphi)\, m^2 + \cos\varphi]\, \eta(t) + \\
& + [(1-\cos\varphi)\, mn + l\sin\varphi]\, \zeta(t)\}, \\
v_y(t) = {} & [(1-\cos\varphi)\, lm - n\sin\varphi]\, u_\xi(t) + \\
& + [(1-\cos\varphi)\, m^2 + \cos\varphi]\, u_\eta(t) + \\
& + [(1-\cos\varphi)\, mn + l\sin\varphi]\, u_\zeta(t) - \\
& - U_\zeta\, \{[(1-\cos\varphi)\, l^2 + \cos\varphi]\, \xi(t) + \\
& + [(1-\cos\varphi)\, ml + n\sin\varphi]\, \eta(t) + \\
& + [(1-\cos\varphi)\, nl - m\sin\varphi]\, \zeta(t)\} + \\
& + U_\xi\, \{[(1-\cos\varphi)\, ln + m\sin\varphi]\, \xi(t) + [(1-\cos\varphi)\, mn - \\
& - l\sin\varphi]\, \eta(t) + [(1-\cos\varphi)\, n^2 + \cos\varphi]\, \zeta(t)\}, \\
v_z(t) = {} & [(1-\cos\varphi)\, ln + m\sin\varphi]\, u_\xi(t) + \\
& + [(1-\cos\varphi)\, mn - l\sin\varphi]\, u_\eta(t) + [(1-\cos\varphi)\, n^2 + \\
& + \cos\varphi]\, u_\zeta(t) - U_\xi\, \{[(1-\cos\varphi)\, lm - n\sin\varphi]\, \xi(t) + \\
& + [(1-\cos\varphi)\, m^2 + \cos\varphi]\, \eta(t) + \\
& + [(1-\cos\varphi)\, mn + l\sin\varphi]\, \zeta(t)\} + \\
& + U_\eta\, \{[(1-\cos\varphi)\, l^2 + \cos\varphi]\, \xi(t) + \\
& + [(1-\cos\varphi)\, ml + n\sin\varphi]\, \eta(t) + \\
& + [(1-\cos\varphi)\, nl - m\sin\varphi]\, \zeta(t)\}.
\end{aligned} \quad (2.3.4)$$

The last equations are somewhat simplified after the replacement in them of projections of angular velocity of the earth U_ξ, U_η and U_ζ respectively by products lU, mU and nU in accordance with the same equalities (2.2.1). As a result we obtain

$$\begin{aligned}
v_x(t) = {} & [(1-\cos\varphi)\, l^2 + \cos\varphi]\, u_\xi(t) + \\
& + [(1-\cos\varphi)\, ml + n\sin\varphi]\, u_\eta(t) + \\
& + [(1-\cos\varphi)\, nl - m\sin\varphi]\, u_\zeta(t) - \\
& - U\, \{\cos\varphi\, [m\zeta(t) - n\eta(t)] - \\
& - l\sin\varphi\, [l\xi(t) + m\eta(t) + n\zeta(t)] + \xi(t)\sin\varphi\}.
\end{aligned}$$

$$v_y(t) = [(1-\cos\varphi)\, lm - n\sin\varphi]\, u_\xi(t) + $$
$$+ [(1-\cos\varphi)\, m^2 + \cos\varphi]\, u_\eta(t) +$$
$$+ [(1-\cos\varphi)\, mn + l\sin\varphi]\, u_\zeta(t) -$$
$$- U\{\cos\varphi\, [n\xi(t) - l\zeta(t)] - m\sin\varphi[l\xi(t) + m\eta(t) +$$
$$+ n\zeta(t)] + \eta(t)\sin\varphi\},$$

$$v_z(t) = [(1-\cos\varphi)\, ln + m\sin\varphi]\, u_\xi(t) +$$
$$+ [(1-\cos\varphi)\, mn - l\sin\varphi]\, u_\eta(t) +$$
$$+ [(1-\cos\varphi)\, n^2 + \cos\varphi]\, u_\zeta(t) -$$
$$- U\{\cos\varphi\, [l\eta(t) - m\xi(t)] -$$
$$- n\sin\varphi\, [l\xi(t) + m\eta(t) + n\zeta(t)] + \zeta(t)\sin\varphi\}. \qquad (2.3.5)$$

It is obvious that by similar means it is possible to arrive at equations which express quantities u_ξ, u_η and u_ζ in terms of v_x, v_y and v_z. We have

$$u_\xi(t) = [(1-\cos\varphi)\, l^2 + \cos\varphi]\, v_x(t) +$$
$$+ [(1-\cos\varphi)\, lm - n\sin\varphi]\, v_y(t) +$$
$$+ [(1-\cos\varphi)\, ln + m\sin\varphi]\, v_z(t) +$$
$$+ U\{\cos\varphi\, [mz(t) - ny(t)] + l\sin\varphi\, [lx(t) +$$
$$+ my(t) + nz(t)] - x(t)\sin\varphi\},$$

$$u_\eta(t) = [(1-\cos\varphi)\, ml + n\sin\varphi]\, v_x(t) +$$
$$+ [(1-\cos\varphi)\, m^2 + \cos\varphi]\, v_y(t) +$$
$$+ [(1-\cos\varphi)\, mn - l\sin\varphi]\, v_z(t) + U\{\cos\varphi[nx(t) -$$
$$- lz(t)] + m\sin\varphi\, [lx(t) + my(t) + nz(t)] - y(t)\sin\varphi\},$$

$$u_\zeta(t) = [(1-\cos\varphi)\, nl - m\sin\varphi]\, v_x(t) +$$
$$+ [(1-\cos\varphi)\, nm + l\sin\varphi]\, v_y(t) +$$
$$+ [(1-\cos\varphi)\, n^2 + \cos\varphi]\, v_z(t) +$$
$$+ U\{\cos\varphi\, [ly(t) - mx(t)] +$$
$$+ n\sin\varphi\, [lx(t) + my(t) + nz(t)] - z(t)\sin\varphi\}. \qquad (2.3.6)$$

§ 4. Error in Flight Range of the Missile as a Function of Changes in Parameters of the End of the Powered-Flight Section in the Nonrotating System of Coordinates. Initial Ballistic Function

We will designate coordinates and projections of velocities of the missile in the nonrotating system of coordinates $\xi\eta\zeta$ at the instant of the end of the powered-flight section of its flight $t = t$, respectively, by letters ξ, η, ζ and u_ξ, u_η and u_ζ. They

are similar to designations (2.1.1) introduced earlier for quantities x, y, z and v_x, v_y, v_z, which refer to the starting system xyz. Thus,

$$\begin{aligned} &\xi(\sigma)=\xi, \quad \eta(\sigma)=\eta, \quad \zeta(\sigma)=\zeta, \\ &u_\xi(\sigma)=u_\xi, \quad u_\eta(\sigma)=u_\eta, \quad u_\zeta(\sigma)=u_\zeta. \end{aligned} \tag{2.4.1}$$

Setting in equations (2.2.5) and (2.3.4) $\phi = U\sigma$, we obtain expressions of quantities x, y, z in terms of ξ, η, ζ and σ and also quantities v_x, v_y, v_z in terms of u_ξ, u_η, u_ζ, ξ, η, ζ and σ. This allows considering magnitude of range l as a function of values of coordinates of the missile $\xi(t)$, $\eta(t)$ and $\zeta(t)$ at the instant $t = \sigma$. projections of its velocity $u_\xi(t)$, $u_\eta(t)$ and $u_\zeta(t)$ in the nonrotational system of coordinates $\xi\eta\zeta$ (at this instant of time) and, finally, the duration itself of the powered-flight section σ. In accordance with formula (2.1.2) we now have

$$l = l(x, y, z, v_x, v_y, v_z) = l(\xi, \eta, \zeta, u_\xi, u_\eta, u_\zeta; \sigma). \tag{2.4.2}$$

It is easy to explain why in this case the range l clearly depends on variable σ. Actually, the position of the rocket relative to the earth at the same values of its coordinates in the nonrotating system $\xi\eta\zeta$ substantially depends on the position of the latter with respect to starting system of coordinates xyz connected to the earth, i.e., on angle $\phi = U\sigma$. The same refers to magnitudes of projections of the vector of velocity of the missile in the starting system at the instant of termination of the powered-flight section.

Calculated values of parameters of the end of the powered-flight section in the starting system of coordinates x^*, y^*, z^*, v_x^*, v_y^*, and v_z^* correspond to calculation values of parameters ξ^*, η^*, ζ^*, u_ξ^*, u_η^*, and u_ζ^* in the nonrotating system $\xi\eta\zeta$. They are connected with each other by equations (2.2.5) and (2.3.4) or (2.2.4) and (2.3.6), in which it follows to assume $\phi = U\sigma^*$, considering σ^* as the designation of the calculated duration of the powered-flight section of flight of the missile. It is obvious that in accordance with formula (2.1.3)

$$l^* = l(x^*, y^*, z^*, v_x^*, v_y^*, v_z^*) = l(\xi^*, \eta^*, \zeta^*, u_\xi^*, u_\eta^*, u_\zeta^*; \sigma^*), \qquad (2.4.3)$$

where l^*, as previously, is the rated value of the range.

The difference Δl between the magnitude of the actual range of the missile l and its calculated value l^* can be represented, similar to equation (2.1.5), in the form of the expansion

$$\Delta l = (\xi - \xi^*)\frac{\partial l}{\partial \xi} + (\eta - \eta^*)\frac{\partial l}{\partial \eta} + (\zeta - \zeta^*)\frac{\partial l}{\partial \zeta} + (u_\xi - u_\xi^*)\frac{\partial l}{\partial u_\xi} + \\ + (u_\eta - u_\eta^*)\frac{\partial l}{\partial u_\eta} + (u_\zeta - u_\zeta^*)\frac{\partial l}{\partial u_\zeta} + (\sigma - \sigma^*)\frac{\partial l}{\partial \sigma}, \qquad (2.4.4)$$

where terms of the second and higher order relative to differences are dropped

$$\xi - \xi^*, \eta - \eta^*, \zeta - \zeta^*, u_\xi - u_\xi^*, u_\eta - u_\eta^*, u_\zeta - u_\zeta^*, \sigma - \sigma^*.$$

In contrast to the expansion (2.1.5), entering into equation (2.4.4) is a term of the first order, which contains by a factor the difference $\sigma - \sigma^*$ between the actual time of the powered-flight section of flight of the missile σ and its calculated value σ^*. The same difference is contained by a number of terms of expansion (2.4.4) of a higher order. The reason for this fact was already explained in the beginning of this section.

Derivatives $\frac{\partial l}{\partial \xi}, \frac{\partial l}{\partial \eta}, \frac{\partial l}{\partial \zeta}, \frac{\partial l}{\partial u_\xi}, \frac{\partial l}{\partial u_\eta}, \frac{\partial l}{\partial u_\zeta}$ and $\frac{\partial l}{\partial \sigma}$ are themselves functions of variables ξ, η, ζ, u_ξ, u_η, u_ζ and σ. In the expansion of (2.4.4) they should be taken at calculated values of enumerated variables ξ^*, η^*, ζ^*, u_ξ^*, u_η^*, u_ζ^*, σ^*, and, consequently, for the specific assigned flight of the missile are constant quantities - ballistic coefficient, which refer to the nonrotating system of coordinates. Let us show that they are all expressed in terms of ballistic coefficients of the starting system $\frac{\partial l}{\partial x}, \frac{\partial l}{\partial y}, \frac{\partial l}{\partial z}, \frac{\partial l}{\partial v_x}, \frac{\partial l}{\partial v_y}$ and $\frac{\partial l}{\partial v_z}$. Actually, according to rules of differentiation of complex functions, specifically, we have:

$$\frac{\partial l}{\partial \xi} = \frac{\partial l}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial l}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial l}{\partial z}\frac{\partial z}{\partial \xi} + \frac{\partial l}{\partial v_x}\frac{\partial v_x}{\partial \xi} + \frac{\partial l}{\partial v_y}\frac{\partial v_y}{\partial \xi} + \frac{\partial l}{\partial v_z}\frac{\partial v_z}{\partial \xi},$$

$$\frac{\partial l}{\partial u_\xi} = \frac{\partial l}{\partial v_x}\frac{\partial v_x}{\partial u_\xi} + \frac{\partial l}{\partial v_y}\frac{\partial v_y}{\partial u_\xi} + \frac{\partial l}{\partial v_z}\frac{\partial v_z}{\partial u_\xi}. \qquad (2.4.5)$$

Analogous expressions take place for derivatives $\frac{\partial l}{\partial \eta}, \frac{\partial l}{\partial \zeta}, \frac{\partial l}{\partial u_\eta}, \frac{\partial l}{\partial u_\zeta}$, and for the partial derivative $\frac{\partial l}{\partial \sigma}$ equation

$$\frac{\partial l}{\partial \sigma} = \frac{\partial l}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial l}{\partial y}\frac{\partial y}{\partial \sigma} + \frac{\partial l}{\partial z}\frac{\partial z}{\partial \sigma} + \frac{\partial l}{\partial v_x}\frac{\partial v_x}{\partial \sigma} + \frac{\partial l}{\partial v_y}\frac{\partial v_y}{\partial \sigma} + \frac{\partial l}{\partial v_z}\frac{\partial v_z}{\partial \sigma}. \qquad (2.4.6)$$

is valid. Entering into enumerated expressions, the partial derivatives of variables x, y, z, v_x, v_y v_z with respect to variables ξ, η, ζ, v_ξ, v_η, v_ζ and σ are formed by means of the application of appropriate operations of differential equations of the type (2.2.5) and (2.3.5), in which it follows preliminarily to set $t = \sigma$ and $\phi = U\sigma$.

Thus, for instance,

$$\frac{\partial x}{\partial \xi} = \frac{\partial v_x}{\partial u_\xi} = \cos U\sigma + (1 - \cos U\sigma)\, l^2,$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial v_x}{\partial u_\eta} = (1 - \cos U\sigma)\, lm + n \sin U\sigma,$$

$$\frac{\partial v_x}{\partial \xi} = -U(1 - l^2) \sin U\sigma,$$

$$\frac{\partial x}{\partial \sigma} = U[-(1 - l^2)\xi \sin U\sigma + (lm \sin U\sigma + n \cos U\sigma)\eta + (ln \sin U\sigma - m \cos U\sigma)\zeta],$$

$$\frac{\partial v_x}{\partial \sigma} = U[-(1 - l^2) u_\xi \sin U\sigma + (lm \sin U\sigma + n \cos U\sigma) u_\eta + (ln \sin U\sigma - m \cos U\sigma) u_\zeta] + U^2 [(m\zeta - n\eta) \sin U\sigma + l(l\xi + m\eta + n\zeta) \cos U\sigma - \xi \cos U\sigma]. \qquad (2.4.7)$$

If now in equations (2.4.5), (2.4.6), (2.4.7) and ones similar to them, we assume the values of all variables equal to the calculated values, then thus the desired coefficients in expression (2.4.4) will be completely determined.

We will assume that as a result of the small deviation in the real motion of the missile from the calculated in the expansion of (2.4.4), the terms caused by the so-called lateral deviation in the missile $(\zeta-\zeta^*)\frac{\partial l}{\partial \zeta}$ and $(u_\zeta-u_\zeta^*)\frac{\partial l}{\partial u_\zeta}$, are small in comparison with other terms. As a result the difference between the actual and calculated flight range of the missile Δl can be represented by the approximate equation

$$\Delta l = (\xi-\xi^*)\frac{\partial l}{\partial \xi} + (\eta-\eta^*)\frac{\partial l}{\partial \eta} + (u_\xi-u_\xi^*)\frac{\partial l}{\partial u_\xi} + \\ + (u_\eta-u_\eta^*)\frac{\partial l}{\partial u_\eta} + (\sigma-\sigma^*)\frac{\partial l}{\partial \sigma}, \qquad (2.4.8)$$

where $\frac{\partial l}{\partial \xi}$, $\frac{\partial l}{\partial \eta}$, $\frac{\partial l}{\partial u_\xi}$, $\frac{\partial l}{\partial u_\eta}$ and $\frac{\partial l}{\partial \sigma}$, as was already shown above, are numbers which are completely defined by the selected calculated motion of the missile.

Expression (2.4.8) corresponds to the following ballistic function, which we call the *initial ballistic function:*

$$\varepsilon(t) = [\xi(t)-\xi^*]\frac{\partial l}{\partial \xi} + [\eta(t)-\eta^*]\frac{\partial l}{\partial \eta} + [u_\xi(t)-u_\xi^*]\frac{\partial l}{\partial u_\xi} + \\ + [u_\eta(t)-u_\eta^*]\frac{\partial l}{\partial u_\eta} + (t-\sigma^*)\frac{\partial l}{\partial \sigma}. \qquad (2.4.9)$$

Here $\xi(t)$, $\eta(t)$, $u_\xi(t)$ and $u_\eta(t)$, as before, are moving coordinates and projections of velocity of the missile in the nonrotating system of coordinates $\xi\eta\zeta$.

If the engine of the missile is turned off at any instant $t = \tau$, then on the basis of equations (2.4.1) and (2.4.8) the value of the ballistic function (2.4.9) corresponding to this instant correct to smallness of the second order determines the error in the flight range of the missile. To avoid this error, one should turn off the engine of the missile at the instant when the ballistic function $\varepsilon(t)$ turns into zero. Thus, the necessary instant of the switching off of the engine is determined by the root of equation

$$[\xi(t)-\xi^{\circ}]\frac{\partial l}{\partial \xi}+[\eta(t)-\eta^{\circ}]\frac{\partial l}{\partial \eta}+[u_{\xi}(t)-u_{\xi}^{\circ}]\frac{\partial l}{\partial u_{\xi}}+$$
$$+[u_{\eta}(t)-u_{\eta}^{\circ}]\frac{\partial l}{\partial u_{\eta}}+(t-\sigma^{\circ})\frac{\partial l}{\partial \sigma}=0. \qquad (2.4.10)$$

Equation (2.4.10) will be called the *initial ballistic equation.*

The basic content of subsequent sections of this chapter and also Chapters III and IV consists in the construction of a series of other ballistic functions, with the help of which it is possible as simply as possible to solve the problem about the inertial control of the flight range of the ballistic missiles. They all are obtained by means of certain conversions from function (2.4.9) and different presentations of coordinates and projections of velocity of the missile in the nonrotating system $\xi\eta\zeta$ in terms of current readings of integrators of accelerations.

§ 5. Differential Equations Which Determine Current Coordinates of Motion of the Missile

The most natural, but, as it appears, not the simplest, method of construction of the left side of the initial ballistic equation (2.4.10) aboard the missile is the preliminary obtaining of most moving coordinates $\xi(t)$ and $\eta(t)$ by means of solution of the differential equations, which connect these coordinates with current readings of two newtonmeters. The axes of sensitivity of the latter should maintain the fixed directions, respectively, parallel to axes ξ and η of the nonrotating system of coordinates $\xi\eta\zeta$. Readings of the mentioned newtonmeters are projections $a_{\xi}(t)$ and $a_{\eta}(t)$ of the apparent acceleration of the missile on the axis ξ and η. According to equation (1.4.1) of Chapter I, in this case we have

$$a_{\xi}(t)=w_{\xi}(t)-f_{\xi}, \qquad a_{\eta}(t)=w_{\eta}(t)-f_{\eta}, \qquad (2.5.1)$$

where

$$w_{\xi}(t)=\frac{d^2\xi(t)}{dt^2}, \qquad w_{\eta}(t)=\frac{d^2\eta(t)}{dt^2} \qquad (2.5.2)$$

are projections on axes ξ and η of the real acceleration of the missile relative to the fixed system of coordinates $\xi\eta\zeta$, and

$$f_\xi = f_\xi[\xi(t), \eta(t), \zeta(t); t],$$
$$f_\eta = f_\eta[\xi(t), \eta(t), \zeta(t); t] \qquad (2.5.3)$$

are, respectively, projections of acceleration of the force of the earth's gravity, which depend in evident form not only on coordinate of the missile, but also on time t. For an explanation of the last fact let us note that even at the point with constant coordinates relative to the nonrotating system $\xi\eta\zeta$ the acceleration of the force of gravity is changed with the course of time. Actually, with respect to the system of coordinates $\xi\eta\zeta$ the position of the earth is continuously changed. At the same time the field of gravity of the earth does not have radial symmetry, specifically, as a result of the deviation in its form from a sphere. If the last fact is disregarded and the earth is considered a sphere with radial and symmetric distribution of density, then projections of acceleration of the force of the earth's gravity will be represented by equations

$$f_\xi = -f_0 R^2 \frac{\xi(t)}{\rho^3}, \qquad f_\eta = -f_0 R^2 \frac{\eta(t)}{\rho^3}, \qquad (2.5.4)$$

no longer dependent in evident form on time. Into them f_0 is the value of acceleration of the force of gravity on the earth's surface examined as a sphere of radius R, and

$$\rho = \sqrt{[\xi(t)]^2 + [\eta(t)]^2 + [\zeta(t)]^2} \qquad (2.5.5)$$

is the distance between the missile and center of the earth. Because of the smallness of the lateral deviation in the rocket $\zeta(t)$ in comparison with the radius of the earth R, this coordinate in equation (2.5.5) can be cropped. Correspondingly, it is possible to drop coordinate $\zeta(t)$ in equations (2.5.3), considering the effect of this coordinate on the magnitude of acceleration of the force of gravity unimportant. Then relations (2.5.1) can be considered as a combination of two differential equations

$$\frac{d^2\xi(t)}{dt^2} = a_\xi(t) + f_\xi[\xi(t), \eta(t); t],$$
$$\frac{d^2\eta(t)}{dt^2} = a_\eta(t) + f_\eta[\xi(t), \eta(t); t] \qquad (2.5.6)$$

for the search of desired functions $\xi(t)$ and $\eta(t)$ on the given current readings $a_\xi(t)$ and $a_\eta(t)$ of two newtonmeters.

The direct solution of the system of nonlinear differential equations (2.5.6) aboard the missile for the purpose of seeking the current values of coordinates $\xi(t)$ and $\eta(t)$ it is possible by means of the use of a rather high-speed computer. In the subsequent two chapters the approximation solution to these equations, which in principle can be used in the system of control of the flight range of the ballistic missile in the presence of onboard of its newtonmeters or integrators of accelerations and simplest computers is used.

§ 6. Auxiliary Relation Connecting Magnitudes of Ballistic Coefficients in the Nonrotating System of Coordinates

At different conversions of the initial ballistic equation (2.4.10) one relation proves to be useful, and it connects magnitudes of the ballistic coefficients $\frac{\partial l}{\partial \xi}, \frac{\partial l}{\partial \eta}, \frac{\partial l}{\partial u_\xi}, \frac{\partial l}{\partial u_\eta}$ and $\frac{\partial l}{\partial \sigma}$ in the nonrotating system of coordinates $\xi\eta\zeta$ with projections of velocity u_ξ^*, u_η^* and with projections of acceleration of the force of gravity $f_\xi(\xi^*, \eta^*; \sigma^*)$, $f_\eta(\xi^*, \eta^*; \sigma^*)$, which refer to the instant $t = \sigma^*$ of the termination of the powered-flight section of the calculated motion of the missile.

Let us introduce functions $\xi^*(t)$ and $\eta^*(t)$, which are current coordinates of the missile in its calculated motion on the powered-flight section. They satisfy the totality of differential equations (2.5.6), if in them the current values of projections $a_\xi(t)$ and $a_\eta(t)$ of the apparent acceleration are replaced by their calculated values $a_\xi^*(t)$ and $a_\eta^*(t)$. Thus,

$$\begin{aligned} \frac{d^2\xi^*(t)}{dt^2} &= a_\xi^*(t) + f_\xi[\xi^*(t), \eta^*(t); t], \\ \frac{d^2\eta^*(t)}{dt^2} &= a_\eta^*(t) + f_\eta[\xi^*(t), \eta^*(t); t]. \end{aligned} \tag{2.6.1}$$

Correspondingly, let us introduce functions $\xi^{**}(t)$ and $\eta^{**}(t)$, which are coordinates of the missile in the free-flight section of its calculated motion. Let us assume that in the beginning of motion of the missile in the free-flight section the resistance of the atmosphere is unimportant, and, consequently, the apparent acceleration is absent. In this case one should consider that functions $\xi^{**}(t)$ and $\eta^{**}(t)$ satisfy the totality of differential equations, which is obtained from (2.6.1) if in equations of the latter we drop quantities $a^{*}_{\xi}(t)$ and $a^{*}_{\eta}(t)$ and replace variables $\xi^{*}(t)$ and $\eta^{*}(t)$ respectively by $\xi^{**}(t)$ and $\eta^{**}(t)$. We obtain

$$\begin{aligned} \frac{d^2\xi^{**}(t)}{dt^2} &= f_\xi[\xi^{**}(t), \eta^{**}(t); t], \\ \frac{d^2\eta^{**}(t)}{dt^2} &= f_\eta[\xi^{**}(t), \eta^{**}(t); t]. \end{aligned} \tag{2.6.2}$$

Let us designate the current values of projections on the axes ξ and η of the velocity of the missile in the powered-flight section with its calculated motion by $u^{*}_{\xi}(t)$ and $u^{*}_{\eta}(t)$, and by $u^{**}_{\xi}(t)$ and $u^{**}_{\eta}(t)$ – the corresponding quantities which refer to the free-flight section. We have the evident equalities

$$u^{*}_{\xi}(t) = \frac{d\xi^{*}(t)}{dt}, \qquad u^{*}_{\eta}(t) = \frac{d\eta^{*}(t)}{dt} \tag{2.6.3}$$

and, similar to them

$$u^{**}_{\xi}(t) = \frac{d\xi^{**}(t)}{dt}, \qquad u^{**}_{\eta}(t) = \frac{d\eta^{**}(t)}{dt}. \tag{2.6.4}$$

The powered-flight section of the calculated motion of the missile at the instant $t = \sigma^{*}$ will pass over into the free-flight section, whence it follows that

$$\xi^{*}(\sigma^{*}) = \xi^{**}(\sigma^{*}) = \xi^{*}, \qquad \eta^{*}(\sigma^{*}) = \eta^{**}(\sigma^{*}) = \eta^{*}, \tag{2.6.5}$$

and also

$$u^{*}_{\xi}(\sigma^{*}) = u^{**}_{\xi}(\sigma^{*}) = u^{*}_{\xi}, \qquad u^{*}_{\eta}(\sigma^{*}) = u^{**}_{\eta}(\sigma^{*}) = u^{*}_{\eta}. \tag{2.6.6}$$

According to equations (2.6.1) and equations (2.6.5) for the end of the calculated powered-flight section,[1] we obtain

$$\frac{d^2\xi^*(\sigma^*)}{dt^2} = a_\xi^*(\sigma^*) + f_\xi(\xi^*, \eta^*; \sigma^*),$$
$$\frac{d^2\eta^*(\sigma^*)}{dt^2} = a_\eta^*(\sigma^*) + f_\eta(\xi^*, \eta^*; \sigma^*). \qquad (2.6.7)$$

Similar relations can be obtained on the basis of equations (2.6.2) and the same equations (2.6.5) for the beginning of the calculated free-flight section. They have the following form:

$$\frac{d^2\xi^{**}(\sigma^*)}{dt^2} = f_\xi(\xi^*, \eta^*; \sigma^*), \qquad \frac{d^2\eta^{**}(\sigma^*)}{dt^2} = f_\eta(\xi^*, \eta^*; \sigma^*). \qquad (2.6.8)$$

Let us examine now on the calculated free-flight section of flight of the missile the certain instant $t = t_1$, which follows after the instant of the switching off of the engine $t = \sigma^*$. Correct to smallness of the second order we have expansions in Taylor series for coordinates of the missile ξ_1 and η_1, which refer to this instant, namely:

$$\xi_1 = \xi^{**}(t_1) = \xi^{**}(\sigma^*) + (t_1 - \sigma^*)\frac{d\xi^{**}(\sigma^*)}{dt},$$
$$\eta_1 = \eta^{**}(t_1) = \eta^{**}(\sigma^*) + (t_1 - \sigma^*)\frac{d\eta^{**}(\sigma^*)}{dt}, \qquad (2.6.9)$$

or, taking account equalities (2.6.4), (2.6.5) and (2.6.6),

$$\xi_1 = \xi^* + (t_1 - \sigma^*)u_\xi^*, \qquad \eta_1 = \eta^* + (t_1 - \sigma^*)u_\eta^*. \qquad (2.6.10)$$

Similarly for projections u_ξ^1 and u_η^1 of the calculated velocity of the missile at the mentioned instant $t = t_1$ we have

[1]In designations of the type $d\xi(\sigma)/dt$ it follows, of course, to consider, that from the beginning an operation of differentiation of the appropriate function is produced, and then the value shown in parenthesis is given to the argument.

$$
\begin{aligned}
u_\xi^1 &= u_\xi^{**}(t_1) = u_\xi^{**}(\sigma^*) + (t_1 - \sigma^*)\frac{du_\xi^{**}(\sigma^*)}{dt},\\
u_\eta^1 &= u_\eta^{**}(t_1) = u_\eta^{**}(\sigma^*) + (t_1 - \sigma^*)\frac{du_\eta^{**}(\sigma^*)}{dt}.
\end{aligned} \tag{2.6.11}
$$

On the basis of equations (2.6.4), it is possible here to produce the following replacement:

$$
\frac{du_\xi^{**}(\sigma^*)}{dt} = \frac{d^2\xi^{**}(\sigma^*)}{dt^2}, \qquad \frac{du_\eta^{**}(\sigma^*)}{dt} = \frac{d^2\eta^{**}(\sigma^*)}{dt^2}. \tag{2.6.12}
$$

Taking into account, furthermore, equality (2.6.6), and also relations (2.6.8), we obtain

$$
\begin{aligned}
u_\xi^1 &= u_\xi^* + (t_1 - \sigma^*) f_\xi(\xi^*, \eta^*; \sigma^*),\\
u_\eta^1 &= u_\eta^* + (t_1 - \sigma^*) f_\eta(\xi^*, \eta^*; \sigma^*).
\end{aligned} \tag{2.6.13}
$$

Let us imagine such a motion of the missile in the powered-flight section different from the calculated, as a result of which at the instant $t = t_1$ it proves to be at the point with coordinates ξ_1 and η_1, possessing a velocity the projections of which on axes ξ and η are respectively equal to quantities u_ξ^1 and u_η^1. If at this instant $t = t_1$ the engine of the missile is turned off, then the following free-flight section of its motion, because of the uniqueness of solution to the problem of dynamics completely coincides with the calculated. Actually, both motions are such that at the same instant $t = t_1$ in them positions and velocities of the missile relative to the system of coordinates $\xi\eta\zeta$ coincide. Furthermore, when $t > t_1$ they are subordinated to the same totality of differential equations, which refers to the free-flight section of its motion. Thus, at such a powered-flight section the rocket will not have an error in the range of its flight. Consequently, parameters of the end of the powered-flight section of its motion ξ_1, η_1, u_ξ^1 and u_η^1 should satisfy the initial ballistic equation (2.4.10), i.e.,

$$
\begin{aligned}
&(\xi_1 - \xi^*)\frac{\partial N}{\partial \xi} + (\eta_1 - \eta^*)\frac{\partial N}{\partial \eta} + (u_\xi^1 - u_\xi^*)\frac{\partial N}{\partial u_\xi} +\\
&+ (u_\eta^1 - u_\eta^*)\frac{\partial N}{\partial u_\eta} + (t_1 - \sigma^*)\frac{\partial N}{\partial \sigma} = 0.
\end{aligned} \tag{2.6.14}
$$

Replacing here quantities ξ_1 and η_1 by their expressions according to equations (2.6.10), and u_ξ^1 and u_η^1 in accordance with equations (2.6.13), we obtain after reduction to the common factor $t_1 - \sigma^*$ equality

$$u_\xi^* \frac{\partial l}{\partial \xi} + u_\eta^* \frac{\partial l}{\partial \eta} + f_\xi(\xi^*, \eta^*; \sigma^*) \frac{\partial l}{\partial u_\xi} + \\ + f_\eta(\xi^*, \eta^*; \sigma^*) \frac{\partial l}{\partial u_\eta} + \frac{\partial l}{\partial \sigma} = 0, \qquad (2.6.15)$$

which connects the ballistic coefficients $\frac{\partial l}{\partial \xi}$, $\frac{\partial l}{\partial \eta}$, $\frac{\partial l}{\partial u_\xi}$, $\frac{\partial l}{\partial u_\eta}$ and $\frac{\partial l}{\partial \sigma}$ in the nonrotating system of coordinates $\xi\eta\zeta$ with projections of acceleration of the force of gravity $f_\xi(\xi^*, \eta^*; \sigma^*)$, $f_\eta(\xi^*, \eta^*; \sigma^*)$ and with projections of velocity of the missile u_ξ^* and u_η^* at the calculated instant of the switching off of the engine.

Let us exclude from equality (2.6.15) projections of acceleration of the force of gravity by means of relations (2.6.7) and use equations (2.6.3). As a result let us obtain the new equality

$$\frac{d\xi^*(\sigma^*)}{dt} \frac{\partial l}{\partial \xi} + \frac{d\eta^*(\sigma^*)}{dt} \frac{\partial l}{\partial \eta} + \left[\frac{d^2\xi^*(\sigma^*)}{dt^2} - a_\xi^*(\sigma^*)\right] \frac{\partial l}{\partial u_\xi} + \\ + \left[\frac{d^2\eta^*(\sigma^*)}{dt^2} - a_\eta^*(\sigma^*)\right] \frac{\partial l}{\partial u_\eta} + \frac{\partial l}{\partial \sigma} = 0, \qquad (2.6.16)$$

which will be used in the following section during the conversion of the expression of the initial ballistic equation (2.4.10) to the form convenient for applications.

§ 7. Isochronal Variations of Coordinates and Projections of the Velocity of the Missile. Basic Ballistic Equation

With the juxtaposition of the real and calculated motions of the missiles, let us call the *isochronal variations* of its coordinates the differences

$$\delta\xi(t) = \xi(t) - \xi^*(t), \qquad \delta\eta(t) = \eta(t) - \eta^*(t). \qquad (2.7.1)$$

Similarly, let us introduce isochronal variations of projections of velocity of the missile

$$\delta u_\xi(t) = u_\xi(t) - u_\xi^*(t), \quad \delta u_\eta(t) = u_\eta(t) - u_\eta^*(t) \qquad (2.7.2)$$

and projections of the apparent acceleration

$$\delta a_\xi = a_\xi(t) - a_\xi^*(t), \quad \delta a_\eta = a_\eta(t) - a_\eta^*(t). \qquad (2.7.3)$$

We will consider the enumerated isochronal variations small quantities, the squares and products of which can be neglected. Thereby, it is proposed that the problem of thrust of the engine both in direction and in magnitude is produced to a sufficient degree accurately.

Functions $\xi^*(t)$, $\eta^*(t)$, $u_\xi^*(t)$, $u_\eta^*(t)$, $a_\xi^*(t)$ and $a_\eta^*(t)$ refer to the powered-flight section of the calculated motion of the missile and, consequently, are determined at values of the argument t not exceeding the duration of this section σ^*. At the same time in equations (2.7.1), (2.7.2) and (2.7.3) functions $\xi(t)$, $\eta(t)$, $u_\xi(t)$, $u_\eta(t)$, $a_\xi(t)$ and $a_\eta(t)$ refer to the powered-flight section of the real motion of the missile, the duration σ of which can be both less and more than the calculated value σ^*. Therefore, for the complete certainty of variations $\delta\xi(t)$, $\delta\eta(t)$, $\delta u_\xi(t)$, $\delta u_\eta(t)$, $\delta a_\xi(t)$ and $\delta a_\eta(t)$ in the whole time interval of the powered-flight section of the real motion of the missile, one should agree upon what is understood by functions $\xi^*(t)$, $\eta^*(t)$, $u_\xi^*(t)$, $u_\eta^*(t)$, $a_\xi^*(t)$, $a_\eta^*(t)$ with argument t somewhat exceeding quantity σ^* (within limits of the allowed variance of duration of the powered-flight section for the specific type of missile). If one assumes that the thrust of the engine with termination of the powered-flight section is lowered gradually or by stages, then for these functions when $t > \sigma^*$ it is natural to take, respectively, functions $\xi^{**}(t)$, $\eta^{**}(t)$, $u_\xi^{**}(t)$ and $u_\eta^{**}(t)$, which refer to the calculated free-flight section and assume functions $a_\xi^*(t)$ and $a_\eta^*(t)$ equal to zero. In this case it follows to expect that the isochronal variations (2.7.1), (2.7.2) and (2.7.3) will be small at the short time interval directly following the calculated instant of the switching off of the engine.

For missiles with the sudden switching off of the engine, on the contrary, it is more preferable to consider functions $\xi^*(t)$, $\eta^*(t)$, $u_\xi^*(t)$ and $u_\eta^*(t)$ when $t > \sigma^*$ equal to their calculated values calculated on the assumption that the engine of the missile at the instant $t = \sigma^*$ was not switched off. Here the isochronal variations (2.7.1), (2.7.2) and (2.7.3) again can be considered small.

Let us use now equations (2.7.1) and (2.7.2) for isochronal variations of coordinates and projections of the velocity in order to transform the initial ballistic equation (2.4.10) by which, as was already mentioned, the necessary time of cessation of operation of the engine of the missile for its hitting of the assigned target is determined.

Let us present in the beginning equation (2.4.10) in the following manner:

$$\begin{aligned}
&[\xi(t)-\xi^*(t)]\frac{\partial l}{\partial \xi}+[\eta(t)-\eta^*(t)]\frac{\partial l}{\partial \eta}+[u_\xi(t)-u_\xi^*(t)]\frac{\partial l}{\partial u_\xi}+\\
&+[u_\eta(t)-u_\eta^*(t)]\frac{\partial l}{\partial u_\eta}=-\Big\{[\xi^*(t)-\xi^*]\frac{\partial l}{\partial \xi}+\\
&+[\eta^*(t)-\eta^*]\frac{\partial l}{\partial \eta}+[u_\xi^*(t)-u_\xi^*]\frac{\partial l}{\partial u_\xi}+\\
&+[u_\eta^*(t)-u_\eta^*]\frac{\partial l}{\partial u_\eta}+(t-\sigma^*)\frac{\partial l}{\partial \sigma}\Big\}. \qquad (2.7.4)
\end{aligned}$$

Into the left side of the last equation let us substitute expressions for isochronal variations of coordinates and projections of the velocity of the rocket, according to equations (2.6.5), (2.7.1) and (2.7.2). In its right side let us expand each of the functions $\xi^*(t)$, $\eta^*(t)$, $u_\xi^*(t)$, $u_\eta^*(t)$ in Taylor series near the value $t = \sigma^*$, retaining in the expansion only terms of the first order of smallness. We obtain

$$\begin{aligned}
&\delta\xi(t)\frac{\partial l}{\partial \xi}+\delta\eta(t)\frac{\partial l}{\partial \eta}+\delta u_\xi(t)\frac{\partial l}{\partial u_\xi}+\delta u_\eta(t)\frac{\partial l}{\partial u_\eta}=\\
&=-(t-\sigma^*)\left[\frac{d\xi^*(\sigma^*)}{dt}\frac{\partial l}{\partial \xi}+\frac{d\eta^*(\sigma^*)}{dt}\frac{\partial l}{\partial \eta}+\right.\\
&\left.+\frac{d^2\xi^*(\sigma^*)}{dt^2}\frac{\partial l}{\partial u_\xi}+\frac{d^2\eta^*(\sigma^*)}{dt^2}\frac{\partial l}{\partial u_\eta}+\frac{\partial l}{\partial s}\right]. \qquad (2.7.5)
\end{aligned}$$

The right side of the last equation can be considerably simplified if we take into account equality (2.6.16) of the previous section. As a result we arrive at the following equation:

$$\begin{aligned}\delta\xi(t)\frac{\partial N}{\partial\xi}+\delta\eta(t)\frac{\partial N}{\partial\eta}+\delta v_{\xi}(t)\frac{\partial N}{\partial v_{\xi}}+\delta v_{\eta}(t)\frac{\partial N}{\partial v_{\eta}}=\\ =-(t-\sigma^{*})\left[a_{\xi}^{*}(\sigma^{*})\frac{\partial N}{\partial v_{\xi}}+a_{\eta}^{*}(\sigma^{*})\frac{\partial N}{\partial v_{\eta}}\right],\end{aligned} \qquad (2.7.6)$$

which, subsequently, we will call the *basic ballistic equation.*

For the actual use of equation (2.7.6) in the system of inertial control of the flight range of ballistic missiles its left side should be constructed aboard the missile at current readings of newtonmeters in the form of a certain electrical or mechanical magnitude.

CHAPTER III

APPROXIMATE PRESENTATIONS OF THE BALLISTIC EQUATION DETERMINING THE INSTANT OF SWITCHING OFF OF THE ENGINE

§ 1. Differential Equations for Isochronal Variations of Coordinates of the Missile in the Powered-Flight Section of Its Motion

In the previous chapter there was obtained the so-called basic ballistic equation (2.7.6), which should satisfy parameters of the end of the powered-flight section in order that the rocket would not have an error in the range of its flight. For the construction of the left side of this equation, it is necessary to know aboard the missile the isochronal variations of coordinates $\delta\xi(t)$ and $\delta\eta(t)$, and also projections of the velocity $\delta u_{\xi}(t)$ and $\delta u_{\eta}(t)$. The latter are, of course, time derivatives of variations of coordinates, i.e.,

$$\delta u_{\xi}(t)=\frac{d\delta\xi(t)}{dt},\qquad \delta u_{\eta}(t)=\frac{d\delta\eta(t)}{dt}. \tag{3.1.1}$$

The coordinates themselves $\xi(t)$ and $\eta(t)$ of the real motion of the missile in the powered-flight section satisfy the totality of differential equations (2.5.6), and coordinates of the calculated motion - totality (2.6.1). Let us form the differences, respectively of the left and right sides of these equations and equate them to each other. Taking into account equalities (2.7.1) and (2.7.3), we obtain

$$\frac{d^2\delta\xi(t)}{dt^2} = \delta a_\xi(t) + f_\xi[\xi^*(t) + \delta\xi(t), \eta^*(t) + \delta\eta(t); t] -$$
$$- f_\xi[\xi^*(t), \eta^*(t); t],$$

$$\frac{d^2\delta\eta(t)}{dt^2} = \delta a_\eta(t) + f_\eta[\xi^*(t) + \delta\xi(t), \eta^*(t) + \delta\eta(t); t] -$$
$$- f_\eta[\xi^*(t), \eta^*(t); t]. \qquad (3.1.2)$$

In the right sides of the last equalities we can use the expansion in the Taylor series of the function according to increases in two of its arguments. Retaining only terms of expansion of the first order relative to variations $\delta\xi(t)$ and $\delta\eta(t)$ and carrying out simplifications, we will arrive at the following totality of two linear differential equations:

$$\frac{d^2\delta\xi(t)}{dt^2} = \delta a_\xi(t) + \frac{\partial f_\xi}{\partial \xi}\delta\xi(t) + \frac{\partial f_\xi}{\partial \eta}\delta\eta(t),$$
$$\frac{d^2\delta\eta(t)}{dt^2} = \delta a_\eta(t) + \frac{\partial f_\eta}{\partial \xi}\delta\xi(t) + \frac{\partial f_\eta}{\partial \eta}\delta\eta(t). \qquad (3.1.3)$$

The desired functions of these equations are isochronal variations of coordinates of the rocket $\delta\xi(t)$ and $\delta\eta(t)$. Initial conditions for them can be obtained from the fact that at the instant of the launch, i.e., when $t = 0$, coordinates and velocities of the missile are the same both in real and in calculated motions. Consequently, according to equations (2.7.1) and (2.7.2), we have

$$\delta\xi(0) = 0, \qquad \delta\eta(0) = 0 \qquad (3.1.4)$$

and further

$$\delta u_\xi(0) = \frac{d\delta\xi(0)}{dt} = 0, \qquad \delta u_\eta(0) = \frac{d\delta\eta(0)}{dt} = 0. \qquad (3.1.5)$$

The right sides of equations (3.1.3) contain variables – isochronal variations of projections of the apparent acceleration of the rocket $\delta a_\xi(t)$ and $\delta a_\eta(t)$. In accordance with equations (2.7.3), in principle they can be obtained aboard the missile by means of a continuous formation of the difference between the real reading of the appropriate newtonmeter and its current calculated value.

Coefficients of equations (3.1.3)[1]

$$\frac{\partial f_\xi}{\partial \xi}, \quad \frac{\partial f_\xi}{\partial \eta} = \frac{\partial f_\eta}{\partial \xi}, \quad \frac{\partial f_\eta}{\partial \eta} \tag{3.1.6}$$

should be considered as time functions, known earlier for each specific case of flight of the missile. In order to obtain these coefficients, in accordance with rules of expansion in Taylor series, in partial derivatives of functions $f_\xi(\xi, \eta; t)$ and $f_\eta(\xi, \eta; t)$ according to variables ξ and η, it is necessary to produce replacement of the latter by functions $\xi^*(t)$ and $\eta^*(t)$. Functions $\xi^*(t)$ and $\eta^*(t)$ are earlier known according to calculated motions of the missile in the powered-flight section. Thus, if aboard the missile continuous (without substantial lag) integration of equations (3.1.3) is carried out and thus magnitudes of functions $\delta\xi(t)$, $\delta\eta(t)$, $\delta u_\xi(t)$ and $\delta u_\eta(t)$ become known, then the left side of the basic ballistic equation (2.7.6) can be constructed by means of multiplying and adding devices. Similarly, with the use of a clock mechanism the right side of this equation is constructed.

§ 2. Approximate Expressions for Isochronal Variations of Coordinates and Projections of Velocity of the Missile. Simplification of the Ballistic Equation

The simplest method of the use of integrators of accelerations in the system of control of the flight range of ballistic missiles is based on the simplification of differential equations (3.1.3). In them terms containing as factors the isochronal variations themselves of coordinates $\delta\xi(t)$ and $\delta\eta(t)$ are dropped. This is equivalent to the assumption about the fact that the effect on the missile of forces of gravity with its actual and calculated motions can be considered practically equal. Such an assumption is admissible only

[1]The equality in (3.1.6) follows from the fact that projections of acceleration of the force of gravity are partial derivatives of the potential of gravity according to appropriate coordinates.

with sufficiently small deviations in the real motion of the missile from the calculated, i.e., at increased requirements for the accuracy of motion of the missile according to the assigned program. Equations (3.1.3) are replaced in this case by the approximate equalities

$$\frac{d^2\delta\xi(t)}{dt^2} = \delta a_\xi(t), \qquad \frac{d^2\delta\eta(t)}{dt^2} = \delta a_\eta(t), \tag{3.2.1}$$

whence allowing for relations (3.1.1), (1.5.4) and (3.1.5), there are equations for variations of projections of the velocity of the missile

$$\begin{aligned} \delta u_\xi(t) &= \frac{d\delta\xi(t)}{dt} = \int_0^t \delta a_\xi(t)\,dt = \delta V_\xi(t), \\ \delta u_\eta(t) &= \frac{d\delta\eta(t)}{dt} = \int_0^t \delta a_\eta(t)\,dt = \delta V_\eta(t). \end{aligned} \tag{3.2.2}$$

Quantities $\delta V_\xi(t)$ and $\delta V_\eta(t)$, which enter into these equations, can be called isochronal variations of the apparent velocity of the missile. According to equations (2.7.3), they are differences

$$\delta V_\xi(t) = V_\xi(t) - V_\xi^*(t), \quad \delta V_\eta(t) = V_\eta(t) - V_\eta^*(t), \tag{3.2.3}$$

in which

$$V_\xi(t) = \int_0^t a_\xi(t)\,dt, \quad V_\eta(t) = \int_0^t a_\eta(t)\,dt \tag{3.2.4}$$

are projections of the apparent velocity of the missile during its actual motion and

$$V_\xi^*(t) = \int_0^t a_\xi^*(t)\,dt, \qquad V_\eta^*(t) = \int_0^t a_\eta^*(t)\,dt \tag{3.2.5}$$

are calculated values of the same magnitudes.

Integrating in turn expressions (3.2.2) with respect to time, we obtain the following equations for isochronal variations of coordinates of the missile:

$$\delta\xi(t) = \int_0^t\int_0^t \delta a_\xi(t)\,dt^2 = \int_0^t \delta V_\xi(t)\,dt = \delta S_\xi(t),$$
$$\delta\eta(t) = \int_0^t\int_0^t \delta a_\eta(t)\,dt^2 = \int_0^t \delta V_\eta(t)\,dt = \delta S_\eta(t). \tag{3.2.6}$$

Here $\delta S_\xi(t)$ and $\delta S_\eta(t)$ are variations of projections of the apparent path, i.e., differences

$$\delta S_\xi(t) = S_\xi(t) - S_\xi^*(t), \quad \delta S_\eta(t) = S_\eta(t) - S_\eta^*(t), \tag{3.2.7}$$

where

$$S_\xi(t) = \int_0^t V_\xi(t)\,dt, \qquad S_\eta(t) = \int_0^t V_\eta(t)\,dt \tag{3.2.8}$$

are projections of the apparent path of the missile and

$$S_\xi^*(t) = \int_0^t V_\xi^*(t)\,dt, \qquad S_\eta^*(t) = \int_0^t V_\eta^*(t)\,dt \tag{3.2.9}$$

their calculated values.

Let us substitute now expressions (3.2.6) and (3.2.2), respectively, for $\delta\xi(t)$, $\delta\eta(t)$, $\delta u_\xi(t)$ and $\delta u_\eta(t)$ into the basic ballistic equation (2.7.6), which determines the instant of the switching off of the engine. We have

$$\begin{aligned} &\delta S_\xi(t)\frac{\partial I}{\partial \xi} + \delta S_\eta(t)\frac{\partial I}{\partial \eta} + \delta V_\xi(t)\frac{\partial I}{\partial u_\xi} + \delta V_\eta(t)\frac{\partial I}{\partial u_\eta} = \\ &\quad = -(t-\sigma^*)\left[a_\xi^*(\sigma^*)\frac{\partial I}{\partial u_\xi} + a_\eta^*(\sigma^*)\frac{\partial I}{\partial u_\eta}\right]. \end{aligned} \tag{3.2.10}$$

Replacing here variations of projections of the apparent velocity and apparent path by their expressions, according to equations (3.2.3) and (3.2.7), we obtain

$$\begin{aligned} &[S_\xi(t) - S_\xi^*(t)]\frac{\partial I}{\partial \xi} + [S_\eta(t) - S_\eta^*(t)]\frac{\partial I}{\partial \eta} + \\ &\quad + [V_\xi(t) - V_\xi^*(t)]\frac{\partial I}{\partial u_\xi} + [V_\eta(t) - V_\eta^*(t)]\frac{\partial I}{\partial u_\eta} = \\ &\quad = -(t-\sigma^*)\left[a_\xi^*(\sigma^*)\frac{\partial I}{\partial u_\xi} + a_\eta^*(\sigma^*)\frac{\partial I}{\partial u_\eta}\right]. \end{aligned} \tag{3.2.11}$$

The last equation can be considerably simplified, if we note that correct to smallness of the first order inclusively the following expansions in Taylor series take place:

$$\begin{aligned} V_\xi^*(t) &= V_\xi^*(\sigma^*) + (t-\sigma^*)\dot{a}_\xi^*(\sigma^*), \\ V_\eta^*(t) &= V_\eta^*(\sigma^*) + (t-\sigma^*)\dot{a}_\eta^*(\sigma^*). \end{aligned} \tag{3.2.12}$$

By means of these equalities equation (3.2.11) is reduced to the form

$$\begin{aligned} &[S_\xi(t) - S_\xi^*(t)]\frac{\partial l}{\partial \xi} + [S_\eta(t) - S_\eta^*(t)]\frac{\partial l}{\partial \eta} + V_\xi(t)\frac{\partial l}{\partial u_\xi} + \\ &+ V_\eta(t)\frac{\partial l}{\partial u_\eta} = V_\xi^*(\sigma^*)\frac{\partial l}{\partial u_\xi} + V_\eta^*(\sigma^*)\frac{\partial l}{\partial u_\eta}, \end{aligned} \tag{3.2.13}$$

where terms dependent on time in evident form are already absent.

Thus, the basic ballistic equation (2.7.6) can be substituted by the approximate equation (3.2.10) or the equation (3.2.13) equivalent to it. For the construction of the left side of equation (3.2.13), the presence is necessary aboard the missile of two integrators of accelerations, axes of which during the whole powered-flight section of flight should retain directions, parallel respectively to axes ξ and η of the nonrotating system of coordinates $\xi\eta\zeta$. Furthermore, there must be a computer, which includes in its composition two additional elements for the time integration of current readings of the very integrators of accelerations and a special element for the reproduction of the calculated current values of projections of the apparent path $S_\xi^*(t)$ and $S_\eta^*(t)$.

§ 3. Construction of the Ballistic Equation by Means of the Use of Readings of Two Integrators of Accelerations with Special Orientation of Their Axes of Sensitivity

There can be considerable interest in the possibility of the reduction in the number of elements of the computer system of inertial determination of the flight range of the ballistic missile

by means of the proper selection of orientation of axes of sensitivity of integrators of accelerations. It turns out that there can be produced a total of one additional integration of current readings of integrators of accelerations, and not two, as in the method of the construction of the ballistic equation given in the previous paragraph. For this purpose, as will be shown below, axes of sensitivity of integrators of accelerations should be parallel to certain directions fixed in each specific case of flight of the missile (so-called λ- and μ-directions).

Another method of simplification of the system of inertial control of the range with the use no longer of an integrator of accelerations, and the meter of the apparent accelerations with the changing orientation of the axis of sensitivity, is stated in the following paragraph. Subsequent modification of this method is given in § 5 of this chapter.

Thus, let us position the axis of sensitivity of one of the integrators of accelerations in the plane $\xi\eta$ of the nonrotating system of coordinates $\xi\eta\zeta$ at a certain constant angle λ to the axis ξ. The direction itself of the axis of sensitivity of this integrator will be called the λ-direction (Fig. 10).

On the basis of equation (1.6.5), after the replacement in it of letters x, y and ν, respectively, by ξ, η and λ and also angle α by angle λ, we obtain equation

$$V_\lambda(t) = V_\xi(t)\cos\lambda + V_\eta(t)\sin\lambda, \tag{3.3.1}$$

which expresses the projection of the apparent velocity of the missile on the mentioned λ-direction by its projections on axes ξ and η. It is obvious that in accordance with formula (1.4.7)

$$V_\lambda(t) = \int_0^t a_\lambda(t)\,dt, \tag{3.3.2}$$

where $a_\lambda(t)$ – component of apparent acceleration along the same λ-direction.

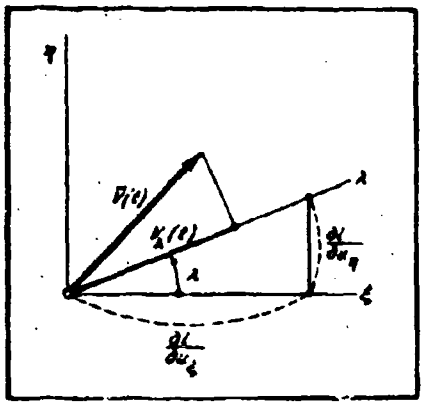

Fig. 10.

Thus, quantity $V_\lambda(t)$ is the current reading of the integrator of accelerations with the axis of sensitivity, oriented in direction λ.

Let us select angle λ so that there would take place equalities

$$\frac{\partial I}{\partial u_\xi} = M\cos\lambda, \qquad \frac{\partial I}{\partial u_\eta} = M\sin\lambda. \tag{3.3.3}$$

Here, as it is easy to see (see Fig. 10), quantity

$$M = \sqrt{\left(\frac{\partial I}{\partial u_\xi}\right)^2 + \left(\frac{\partial I}{\partial u_\eta}\right)^2} \tag{3.3.4}$$

is for each specific case of flight of the missile, an earlier known quantity.

Using equations (3.3.1) and (3.3.3), let us transform the sum of the last two corresponding terms of the left side of equation (3.2.13). We have

$$V_\xi(t)\frac{\partial I}{\partial u_\xi} + V_\eta(t)\frac{\partial I}{\partial u_\eta} = M\,[V_\xi(t)\cos\lambda + V_\eta(t)\sin\lambda] = MV_\lambda(t). \tag{3.3.5}$$

Thus, the reduced sum correct to the constant factor M is determined by the reading of the integrator of accelerations, the

axis of sensitivity of which is located along the λ-direction.

As a result of relation (3.3.5), we have

$$V_\xi^*(\sigma^*)\frac{\partial N}{\partial u_\xi}+V_\eta^*(\sigma^*)\frac{\partial N}{\partial u_\eta}=MV_\lambda^*(\sigma^*). \tag{3.3.6}$$

Similar to relation (3.3.5), we can obtain the following equalities

$$\begin{aligned} S_\xi(t)\frac{\partial N}{\partial \xi}+S_\eta(t)\frac{\partial N}{\partial \eta}&=N\,[S_\xi(t)\cos\mu+S_\eta(t)\sin\mu]=NS_\mu(t)\\ S_\xi^*(t)\frac{\partial N}{\partial \xi}+S_\eta^*(t)\frac{\partial N}{\partial \eta}&=N\,[S_\xi^*(t)\cos\mu+S_\eta^*(t)\sin\mu]=NS_\mu^*(t). \end{aligned} \tag{3.3.7}$$

In them angle μ (Fig. 11) is determined by means of equations

$$\frac{\partial N}{\partial \xi}=N\cos\mu,\qquad \frac{\partial N}{\partial \eta}=N\sin\mu, \tag{3.3.8}$$

and the constant factor N – by equality

$$N=\sqrt{\left(\frac{\partial N}{\partial \xi}\right)^2+\left(\frac{\partial N}{\partial \eta}\right)^2}. \tag{3.3.9}$$

In right sides of equations (3.3.7) $S_\mu(t)$ – projection of the vector of the apparent path of the missile on the so-called μ-direction, which forms the constant angle μ with axis ξ (Fig. 11), and $S_\mu^*(t)$ – current calculated value of this projection. By analogy with equations (1.6.7) and (1.4.7),

$$S_\mu(t)=\int_0^t V_\mu(t)\,dt=\iint_0^{t\,t} a_\mu(t)\,dt^2. \tag{3.3.10}$$

Let us turn now again to equation (3.2.13). Taking into account equalities (3.3.5), (3.3.6) and (3.3.7), we obtain

$$MV_\lambda(t)+N\,[S_\mu(t)-S_\mu^*(t)]=MV_\lambda^*(\sigma^*). \tag{3.3.11}$$

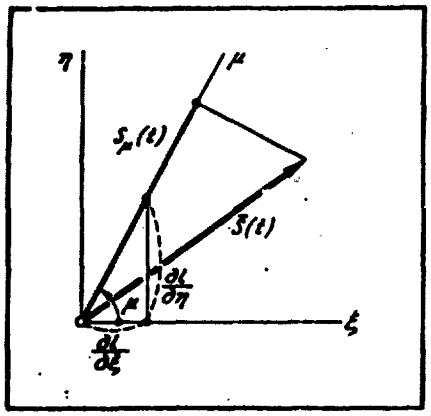

Fig. 11.

The right side of the last equation is a constant determined for each specific case of flight of the missile. This magnitude correct to the constant factor M is equal to the calculated value of projection on the λ-direction of the vector of apparent velocity of the missile at the calculated instant of the switching off of its engine $t = \sigma^*$. If in equation (3.3.11) we introduce another designation[1]

$$p = \frac{N}{M} = \sqrt{\frac{\left(\frac{\partial l}{\partial \xi}\right)^2 + \left(\frac{\partial l}{\partial \eta}\right)^2}{\left(\frac{\partial l}{\partial u_\xi}\right)^2 + \left(\frac{\partial l}{\partial u_\eta}\right)^2}}, \qquad (3.3.12)$$

then it can be reduced to the following final form:

$$V_\lambda(t) + p\left[S_\mu(t) - S_\mu^*(t)\right] = V_\lambda^*(\sigma^*). \qquad (3.3.13)$$

For the construction aboard the missile of the left side of the ballistic equation (3.3.13), an additional integration of current readings of only one integrator of accelerations is necessary with the axis of sensitivity oriented in the μ-direction. Furthermore, the reproduction of the calculated values $S_\mu^*(t)$ of this repeated integration is necessary.

[1]Let us note that coefficient p, similar to factor $1/\tau$ in equation (1.6.11), has a dimensionality opposite to time (in particular s^{-1}).

§ 4. Construction of the Ballistic Equation with the Help of a Special Meter of Apparent Accelerations of the Computer

Let us examine another conversion of the ballistic equation (3.2.13), as a result of which the possibility of solution to the problem of inertial control of flight range of the ballistic missile by means of a single newtonmeter is explained. Its readings in the computer should be integrated after preliminary multiplication by a certain assigned time function. The axis of sensitivity of the newtonmeter in turn change its orientation of the nonrotating system of coordinates according to the assigned law.

Let us turn, first of all, to the conversion of the first two corresponding terms of the ballistic equations (3.2.13). According to equations (3.2.4) and (3.2.8) and also the known Cauchy equation of the conversion of the repeated integral into a simple one containing the upper threshold in the form of the parameter in the subintegral expression, we have

$$S_\xi(t) = \int_0^t V_\xi(t)\,dt = \int_0^t\int_0^t a_\xi(t)\,dt^2 = \int_0^t (t-\tau)\,a_\xi(\tau)\,d\tau, \tag{3.4.1}$$

where τ — new variable of integration. Representing here the difference $t - \tau$ in the form

$$t - \tau = (t - \sigma^*) + (\sigma^* - \tau), \tag{3.4.2}$$

we obtain equality

$$S_\xi(t) = (t-\sigma^*)\int_0^t a_\xi(\tau)\,d\tau + \int_0^t (\sigma^*-\tau)\,a_\xi(\tau)\,d\tau. \tag{3.4.3}$$

Returning in the last integral of the right side of this equality to the initial variable of integration t and taking into account equation (3.2.4), we obtain

$$S_\xi(t) = (t-\sigma^*)V_\xi(t) + \int_{\sigma^*}^{t} (\sigma^* - t)\, a_\xi(t)\, dt. \tag{3.4.4}$$

Let us note further that correct to smallness of the first order inclusively we can assume that

$$S_\xi^*(t) = S_\xi^*(\sigma^*) + (t-\sigma^*)V_\xi^*(\sigma^*). \tag{3.4.5}$$

Equating to each other differences of the left and right sides of the last two equalities, we have

$$\begin{aligned} S_\xi(t) - S_\xi^*(t) = \int_{\sigma^*}^{t} (\sigma^* - t)\, a_\xi(t)\, dt - S_\xi^*(\sigma^*) + \\ + (t-\sigma^*)[V_\xi(t) - V_\xi^*(\sigma^*)]. \end{aligned} \tag{3.4.6}$$

The last term of the right side of equality (3.4.6) has a second order of smallness and can be omitted. Actually, as a result of the proximity of the real and calculated instants $t = \sigma$ and $t = \sigma^*$ of the switching off of the engine quantity $V_\xi(t)$ differs little from $V_\xi(\sigma^*)$, and this latter is distinguished from $V_\xi^*(\sigma^*)$ by a small isochronal variation $\delta V_\xi^*(\sigma^*)$. Therefore the difference $V_\xi(t) - V_\xi^*(\sigma^*)$ is a small magnitude, which in equation (3.4.6) is multiplied in turn by another small difference $t - \sigma^*$.

Thus, correct to smallness of the second order

$$S_\xi(t) - S_\xi^*(t) = \int_{\sigma^*}^{t} (\sigma^* - t)\, a_\xi(t)\, dt - S_\xi^*(\sigma^*) \tag{3.4.7}$$

in perfect analogy

$$S_\eta(t) - S_\eta^*(t) = \int_{\sigma^*}^{t} (\sigma^* - t)\, a_\eta(t)\, dt - S_\eta^*(\sigma^*). \tag{3.4.8}$$

Substituting differences (3.4.7) and (3.4.8) into equation (3.2.13) and replacing in it, furthermore, quantities $V_\xi(t)$ and $V_\eta(t)$ by their representations, according to equations (3.2.4), we obtain after simplifications

$$\int_0^{\sigma^*}\left\{\left[\frac{\partial I}{\partial u_\xi}+(\sigma^*-t)\frac{\partial I}{\partial \xi}\right]a_\xi(t)+\left[\frac{\partial I}{\partial u_\eta}+(\sigma^*-t)\frac{\partial I}{\partial \eta}\right]a_\eta(t)\right\}dt=$$
$$=\frac{\partial I}{\partial u_\xi}V_\xi^*(\sigma^*)+\frac{\partial I}{\partial u_\eta}V_\eta^*(\sigma^*)+\frac{\partial I}{\partial \xi}S_\xi^*(\sigma^*)+\frac{\partial I}{\partial \eta}S_\eta^*(\sigma^*). \qquad (3.4.9)$$

The right side of the last equation for the specific flight of the rocket is a certain constant, which we designate by the letter C. In accordance with equations similar to (3.3.6) and (3.3.7), this constant can also be represented in the form

$$C=\frac{\partial I}{\partial u_\xi}V_\xi^*(\sigma^*)+\frac{\partial I}{\partial u_\eta}V_\eta^*(\sigma^*)+\frac{\partial I}{\partial \xi}S_\xi^*(\sigma^*)+\frac{\partial I}{\partial \eta}S_\eta^*(\sigma^*)=$$
$$=MV_\lambda^*(\sigma^*)+NS_\mu^*(\sigma^*). \qquad (3.4.10)$$

Conversion of the integrand expression of the left side of equation (3.4.9) can be produced by the same methods which were incorporated in the previous section. Let us present in this equation alternating factors before projections of the apparent accelerations $a_\xi(t)$ and $a_\eta(t)$ in the following manner:

$$\begin{aligned}\frac{\partial I}{\partial u_\xi}+(\sigma^*-t)\frac{\partial I}{\partial \xi}&=K(t)\cos\varkappa(t),\\ \frac{\partial I}{\partial u_\eta}+(\sigma^*-t)\frac{\partial I}{\partial \eta}&=K(t)\sin\varkappa(t).\end{aligned} \qquad (3.4.11)$$

In accordance with the last equalities,

$$K(t)=\sqrt{\left[\frac{\partial I}{\partial u_\xi}+(\sigma^*-t)\frac{\partial I}{\partial \xi}\right]^2+\left[\frac{\partial I}{\partial u_\eta}+(\sigma^*-t)\frac{\partial I}{\partial \eta}\right]^2},$$
$$\operatorname{tg}\varkappa(t)=\frac{\frac{\partial I}{\partial u_\eta}+(\sigma^*-t)\frac{\partial I}{\partial \eta}}{\frac{\partial I}{\partial u_\xi}+(\sigma^*-t)\frac{\partial I}{\partial \xi}}. \qquad (3.4.12)$$

The second equation (3.4.12) determines $\varkappa(t)$ changing with time angle between the so-called $\varkappa$-direction and axis ξ (Fig. 12).

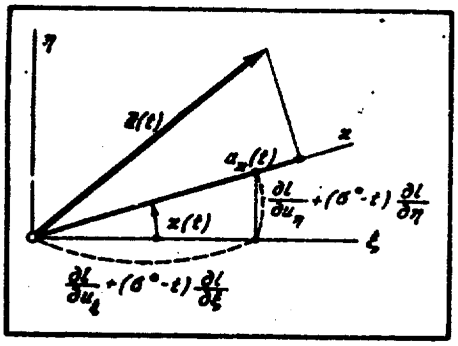

Fig. 12.

Using equations (3.3.3), (3.3.8) and (3.3.12), expression (3.4.12) for variable $K(t)$, after comparatively simple conversions, can also be presented in the form

$$K(t) = M\sqrt{1 + 2p(\sigma^{\circ} - t)\cos(\mu - \lambda) + p^{2}(\sigma^{\circ} - t)^{2}}. \qquad (3.4.13)$$

Let us produce in the left side of equation (3.4.9) replacement of coefficients according to equalities (3.4.10) and (3.4.11). We obtain the relation

$$\int_{0}^{t} K(t)\,[a_{\xi}(t)\cos\varkappa(t) + a_{\eta}(t)\sin\varkappa(t)]\,dt = C, \qquad (3.4.14)$$

where in integrand expression the sum

$$a_{\xi}(t)\cos\varkappa(t) + a_{\eta}(t)\sin\varkappa(t) = a_{\varkappa}(t) \qquad (3.4.15)$$

is the projection of the apparent acceleration of the missile toward the mentioned variable κ-direction (Fig. 12), and constant C is expressed by equation (3.4.10). As a result we obtain the ballistic equation in the following form:

$$\int_{0}^{t} K(t)\,a_{\varkappa}(t)\,dt = C. \qquad (3.4.16)$$

Construction of the left side of equation (3.4.16) aboard the missile requires the presence of highly accurate follow-up systems

located on a gyroscopic stabilizer for the change in orientation of the axis of sensitivity of the meter of acceleration according to the assigned program.

Of course, approximation methods of the formation of the left side of the ballistic equation (3.4.16) by means of the usual integrators of accelerations, for example, by means of replacement of functions $K(t)$ and $\kappa(t)$ by their certain mean values.

§ 5. Inertial Control of Range of the Ballistic Missile by Means of Longitudinal and Standard Integrators of Accelerations

In systems of control of ballistic missiles one can use the so-called standard integrators of apparent accelerations, the axes of sensitivity of which are guided in plane $\xi\eta$ at the assigned angle to the axis ξ according to the program of flight. The control of flight controls is produced in this case so that together with the fulfilling of the assigned program of the change in pitch of the missile (i.e., change in angle between its longitudinal axis and axis ξ) the reading on the standard integrator would be reduced to zero. The angle between the axis of sensitivity of the standard integrator and axis of the missile itself is selected close to a straight line.

The use of the standard integrator of the apparent accelerations in the system of control of motion of the missile leads to stabilization in the assigned direction of the resulting force of thrust of the engine and aerodynamic forces acting on the rocket. Thus, the direction of the vector of the apparent acceleration of the missile (however, of course, not its magnitude) is stabilized. With the known approximation, being distracted from errors of the system of control, it can be considered that the standard integrator controls the flight of the missile so that the projection of the apparent acceleration on the axis of sensitivity of this integrator would be equal to zero.

Let us present in the ballistic equation (3.4.16) of the previous section quantity $a_\kappa(t)$ in the form of the product

$$a_\kappa(t) = a(t)\cos\gamma(t). \qquad (3.5.1)$$

Here $\gamma(t)$ is the angle which forms the vector of the apparent acceleration $a(t)$ with κ-direction inclined at angle $\kappa(t)$ to axis ξ (Fig. 13).

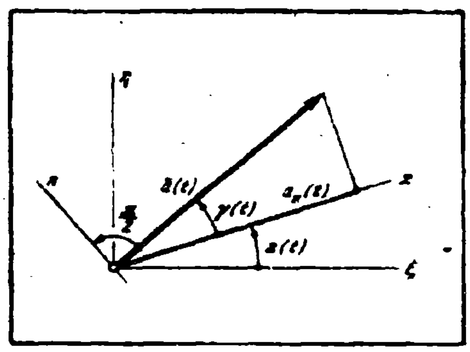

Fig. 13.

Function $\kappa(t)$ is determined by the second equality (3.4.12), and the direction of the apparent acceleration is changed according to the assigned law. Because of this the magnitude of angle $\gamma(t)$ should be considered as a known function of time.

Producing in equation (3.4.16) the replacement of quantity $a_\kappa(t)$, according to equation (3.5.1) we will arrive at the following modification of the ballistic equation:

$$\int_0^t Q(t)a(t)\,dt = C, \qquad (3.5.2)$$

in which the alternating coefficient

$$Q(t) = K(t)\cos\gamma(t) \qquad (3.5.3)$$

should be considered the known time function for the given calculated case of flight of the missile.

For construction of current values of the left side of equation (3.5.2) aboard the missile just as earlier (see § 4 of this chapter), integration of current values of the apparent acceleration, multiplied by the assigned time function is required. However, now there is no need in special high-precision equipment for the change in orientation of the axis of sensitivity of the meter of the apparent acceleration of the missile. Actually, if this newtonmeter is rigidly connected to the standard integrator so that axes of their sensitivity would be perpendicular to each other, then with the proper accuracy of the control of flight of the missile the apparent acceleration of the latter (more accurately, apparent acceleration of location of these instruments) will be wholly directed along the axis of sensitivity of the mentioned newtonmeter.

With some loss of accuracy of flight of the missile the newtonmeter can be fastened directly aboard the missile, the guiding axis of its sensitivity in parallel to the longitudinal axis of the missile. The latter is deflected from the direction of the apparent acceleration of the missile, i.e., from the direction of the resulting force of thrust of the engine and forces of aerodynamic actions, as a rule, at small angles. Because of this, approximately to lay

$$a_0(t) \approx a(t), \qquad (3.5.4)$$

where $a_0(t)$ – projection of apparent acceleration of the missile on its longitudinal axis.

If, in accordance to the last approximate equality, we produce the appropriate replacement in equation (3.5.2) and, furthermore, substitute function $Q(t)$ by its certain mean value Q, then we will obtain the approximate equation

$$Q\int_0^t a_0(t)\,dt = Q\bar{V}_0(t) = C \qquad (3.5.5)$$

for the determination of the instant of switching off of the engine of the missile. Here expression

$$\tilde{V}_0(t)=\int_0^t a_0(t)\,dt \tag{3.5.6}$$

is the current reading of the integrator of acceleration with the axis of sensitivity parallel to the longitudinal axis of the missile.[1] Such an integrator is called longitudinal. As is known, it was used in the guidance system of the German missile V-2.

Let us return again to equation (3.5.2) and examine additionally certain approximate methods of construction of its left side on the basis of current readings

$$\tilde{V}(t)=\int_0^t a(t)\,dt \tag{3.5.7}$$

of the integrator of accelerations with the axis of sensitivity located perpendicular to the axis of the standard integrator. Considering the latter equality the left side of equation (3.5.2) is converted to the form

$$\int_0^t Q(t)\,a(t)\,dt=\int_0^t Q(t)\frac{d\tilde{V}(t)}{dt}\,dt \tag{3.5.8}$$

and can be integrated by parts. As a result, taking into consideration that $\tilde{V}(0) = 0$, we obtain, according to relations (3.5.8) and (3.5.2), the ballistic equation in the following form:

$$Q(t)\tilde{V}(t)-\int_0^t Q'(t)\tilde{V}(t)\,dt=C. \tag{3.5.9}$$

[1]Let us note that quantity $\tilde{V}_0(t)$ in equation (3.5.6) because of the variability of direction of the longitudinal axis is not the projection on this axis of the apparent velocity of the missile in the nonrotating system of coordinates (for more detail on this see § 4 of Chapter I).

The construction aboard the missile of the left side of the last equation can already be produced by means of only one integrator of acceleration.

Let us note in conclusion that the same method of integration by parts can be used for constructing the left side of the ballistic equation (3.4.16).

CHAPTER IV

INERTIAL CONTROL OF THE RANGE OF FLIGHT OF THE BALLISTIC MISSILE ALLOWING FOR THE CHANGE IN THE FORCE OF GRAVITY

§ 1. One Method of the Solution of Differential Equations by Which Isochronal Variations of Coordinates of the Missile with Its Motion in the Powered-Flight Section Are Determined

In the previous chapter the basic ballistic equation (2.7.6) was transformed to such a form at which its left side could be constructed aboard the missile with the help of integrators of accelerations. In this case isochronal variations of coordinates and projections of velocity of the missile, which enter into the composition of equation (2.7.6), were expressed by the approximate equations (3.2.6) and (3.2.2). This corresponded to the neglect in differential equations (3.1.3) of terms containing as factors the isochronal variations $\delta\xi(t)$ and $\delta\eta(t)$ themselves. As was already indicated in Chapter III, the mentioned terms of equations (3.1.3) take into account during the determination of current coordinates of the rocket changes in acceleration of the force of gravity because of the noncoincidence of its actual motion with the calculated. Error appearing from such a simplification of equations (3.1.3) in the determining of the range of flight of the missile is small only at rather small deviations in its real motion from the calculated. Otherwise it is necessary to solve approximately aboard the missile differential equations (3.1.3) allowing for terms reflecting the effect of the change in acceleration of the force of gravity.

Below for the solution of equations (3.1.3) a similar approximation method is used, and it was shown to the author by corresponding member of the Academy of Sciences of the Ukrainian SSR Yu. D. Sokolov.[1] By means of this method the problem of inertial control of range can be quite accurately solved by the means given in Chapter III with the help of the application of standard integrators of accelerations and the simplest computers.

The idea of the method consists in the conversion of equations (3.1.3) to their equivalent integral-differential and integral relations with subsequent replacement under the sign of integral of expressions of desired functions by their simplest approximations, which satisfy, however, initial and final conditions of the problem. In accordance with this, let us integrate the right and left sides of equations (3.1.3) with respect to time from the initial instant $t = 0$ up to the unknown until the instant of switching off of the engine $t = \sigma$, which provides the hitting of the missile on the target. Taking into account in this case initial conditions (3.1.5) for isochronal variations of projections of the velocity of the missile, we obtain two integral-differential relations

$$\begin{aligned} \frac{d\delta\xi(\sigma)}{dt} &= \delta u_\xi(\sigma) = \delta V_\xi(\sigma) + \int_0^\sigma \frac{\partial f_\xi}{\partial \xi}\,\delta\xi(t)\,dt + \int_0^\sigma \frac{\partial f_\xi}{\partial \eta}\,\delta\eta(t)\,dt, \\ \frac{d\delta\eta(\sigma)}{dt} &= \delta u_\eta(\sigma) = \delta V_\eta(\sigma) + \int_0^\sigma \frac{\partial f_\eta}{\partial \xi}\,\delta\xi(t)\,dt + \int_0^\sigma \frac{\partial f_\eta}{\partial \eta}\,\delta\eta(t)\,dt, \end{aligned} \qquad (4.1.1)$$

in which, similar to equalities (3.2.2),

$$\delta V_\xi(\sigma) = \int_0^\sigma \delta a_\xi(t)\,dt, \qquad \delta V_\eta(\sigma) = \int_0^\sigma \delta a_\eta(t)\,dt \qquad (4.1.2)$$

the isochronal variations of projections of the apparent velocity of the missile on axes ξ and η at instant $t = \sigma$.

[1]See, for example, Yu. D. Sokolov. One method of observed solved linear integral differential equations. – Dop. AN URSR, 1955, No. 2.

If, however, the right and left sides of equations (3.1.3) are integrated with respect to time twice: one time within the limits of $t = 0$ to the current instant t and the second - also from $t = 0$ to the instant of switching off of the engine $t = \sigma$, then, taking account initial conditions (3.1.4) and (3.1.5), we obtain an additional two integral relation:

$$\delta\xi(\sigma) = \delta S_\xi(\sigma) + \int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial \xi}\,\delta\xi(t)\,dt^2 + \int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial \eta}\,\delta\eta(t)\,dt^2,$$

$$\delta\eta(\sigma) = \delta S_\eta(\sigma) + \int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial \xi}\,\delta\xi(t)\,dt^2 + \int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial \eta}\,\delta\eta(t)\,dt^2, \qquad (4.1.3)$$

where in turn, in accordance with equalities (3.2.6),

$$\delta S_\xi(\sigma) = \int_0^\sigma \delta V_\xi(t)\,dt = \int_0^\sigma\int_0^t \delta a_\xi(t)\,dt^2,$$

$$\delta S_\eta(\sigma) = \int_0^\sigma \delta V_\eta(t)\,dt = \int_0^\sigma\int_0^t \delta a_\eta(t)\,dt^2 \qquad (4.1.4)$$

isochronal variations of projections of the apparent path of the rocket, which refer to the instant $t = \sigma$.

In accordance with the mentioned method, one should substitute in the right sides of relations (4.1.1) and (4.1.3) functions $\delta\xi(t)$ and $\delta\eta(t)$ by their approximate representations, which turn into zero at the initial instant of time and respectively into $\delta\xi(\sigma)$ and $\delta\eta(\sigma)$ when $t = \sigma$. Then values of quantities $\delta\xi(\sigma)$, $\delta\eta(\sigma)$, $\delta u_\xi(\sigma)$ and $\delta u_\eta(\sigma)$, found in accordance with the mentioned relations, prove to be, as a rule, more accurate than those calculated according to equations (3.2.2) and (3.2.6), founded upon approximate equalities (3.2.1).

It is possible to take as approximate representations $\delta\xi(t)$ and $\delta\eta(t)$ in right sides of relations (4.1.1) and (4.1.3), for example, their linear approximation

$$\delta\xi(t) = \delta\xi(\sigma)\frac{t}{\sigma}, \qquad \delta\eta(t) = \delta\eta(\sigma)\frac{t}{\sigma}, \tag{4.1.5}$$

where σ – instant of time for which the determination of desired functions is produced; in this case σ – instant of termination of the powered-flight section of flight of the missile.

The best results should be expected with the quadratic approximation of the form

$$\delta\xi(t) = \delta\xi(\sigma)\frac{t^2}{\sigma^2}, \qquad \delta\eta(t) = \delta\eta(\sigma)\frac{t^2}{\sigma^2}. \tag{4.1.6}$$

The reason for this is that functions (4.1.6), unlike functions (4.1.5), satisfy simultaneously initial conditions (3.1.4) and (3.1.5), which concern both variations themselves of the coordinates and their time derivatives at the instant $t = 0$. Functions (4.1.5) do not satisfy initial conditions (3.1.5).

Leading to an even greater accuracy should be the assignment of functions $\delta\xi(t)$ and $\delta\eta(t)$ in the form of the following polynomials of the third power

$$\begin{aligned} \delta\xi(t) &= \delta\xi(\sigma)\left(3\frac{t^2}{\sigma^2} - 2\frac{t^3}{\sigma^3}\right) + \delta u_\xi(\sigma)\cdot\sigma\left(\frac{t^3}{\sigma^3} - \frac{t^2}{\sigma^2}\right), \\ \delta\eta(t) &= \delta\eta(\sigma)\left(3\frac{t^2}{\sigma^2} - 2\frac{t^3}{\sigma^3}\right) + \delta u_\eta(\sigma)\cdot\sigma\left(\frac{t^3}{\sigma^3} - \frac{t^2}{\sigma^2}\right). \end{aligned} \tag{4.1.7}$$

These polynomials not only satisfy initial conditions (3.1.4) and (3.1.5), but, furthermore, when $t = \sigma$ turn, respectively, into $\delta\xi(\sigma)$ and $\delta\eta(\sigma)$ and their derivatives – into $\delta u_\xi(\sigma)$ and $\delta u_\eta(\sigma)$.

§ 2. Approximate Solution of Differential Equations for Variations of Coordinates Using the Quadratic Approximation and Also the Approximation in the Form of Polynomials of the Third Power

Let us examine from the beginning the solution of differential equations (3.1.3) which corresponds to the quadratic approximation (4.1.6). Substituting into right sides of relations (4.1.3)

expressions for $\delta\xi(t)$ and $\delta\eta(t)$, according to equations (4.1.6), we obtain

$$\delta\xi(\sigma) = \delta S_\xi(\sigma) + \frac{\delta\xi(\sigma)}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial\xi} t^2\,dt^2 + \frac{\delta\eta(\sigma)}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial\eta} t^2\,dt^2,$$

$$\delta\eta(\sigma) = \delta S_\eta(\sigma) + \frac{\delta\xi(\sigma)}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial\xi} t^2\,dt^2 + \frac{\delta\eta(\sigma)}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial\eta} t^2\,dt^2. \qquad (4.2.1)$$

Let us introduce here designations

$$h_{\xi\xi}(\sigma) = \frac{1}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial\xi} t^2\,dt^2; \quad h_{\xi\eta}(\sigma) = \frac{1}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial\eta} t^2\,dt^2;$$

$$h_{\eta\xi}(\sigma) = \frac{1}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial\xi} t^2\,dt^2; \quad h_{\eta\eta}(\sigma) = \frac{1}{\sigma^2}\int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial\eta} t^2\,dt^2, \qquad (4.2.2)$$

where, because of relations (3.1.6),

$$h_{\xi\eta}(\sigma) = h_{\eta\xi}(\sigma). \qquad (4.2.3)$$

As a result let us arrive at the two algebraic equations:

$$\delta\xi(\sigma) = \delta S_\xi(\sigma) + h_{\xi\xi}(\sigma)\,\delta\xi(\sigma) + h_{\xi\eta}(\sigma)\,\delta\eta(\sigma),$$
$$\delta\eta(\sigma) = \delta S_\eta(\sigma) + h_{\eta\xi}(\sigma)\,\delta\xi(\sigma) + h_{\eta\eta}(\sigma)\,\delta\eta(\sigma) \qquad (4.2.4)$$

with respect to the desired quantities $\delta\xi(\sigma)$ and $\delta\eta(\sigma)$.

Values of coefficients $h_{\xi\xi}(\sigma)$, $h_{\xi\eta}(\sigma) = h_{\eta\xi}(\sigma)$ and $h_{\eta\eta}(\sigma)$ with an accuracy sufficient for practice can be taken with argument $\sigma = \sigma^*$. Actually, the real duration of the powered-flight section σ only by a small magnitude is distinguished from the calculated σ^*. Isochronal variations $\delta\xi(\sigma)$ and $\delta\eta(\sigma)$ should also be considered as small magnitudes. Consequently, the error which occurs from the mentioned replacement in coefficients $h_{\xi\xi}(\sigma)$, $h_{\xi\eta}(\sigma)$ and $h_{\eta\eta}(\sigma)$, which stand in the right side of equations (4.2.4), of the value of argument σ by σ^*, has the second order of smallness.

Let us designate coefficients (4.2.2) when $\sigma = \sigma^*$ by the same letters but without indication of the argument. With the made simplifications and replacement of the designation of argument σ by t equations (4.2.4) take the following form:

$$\begin{aligned} \delta\xi(t) &= \delta S_\xi(t) + h_{\xi\xi}\delta\xi(t) + h_{\xi\eta}\delta\eta(t), \\ \delta\eta(t) &= \delta S_\eta(t) + h_{\eta\xi}\delta\xi(t) + h_{\eta\eta}\delta\eta(t), \end{aligned} \tag{4.2.5}$$

where $h_{\xi\xi}$, $h_{\xi\eta} = h_{\eta\xi}$, $h_{\eta\eta}$ are quantities known for the given calculated flight of the missile, which are small in comparison with unity (see § 3 of this chapter). Because of this the totality of equations (4.2.5) is most convenient to solve by means of iterations, taking $\delta\xi(t)$ and $\delta\eta(t)$ in the right side of equations (4.2.5) the basic expressions represented by equations (3.2.6), namely

$$\delta\xi(t) = \delta S_\xi(t), \qquad \delta\eta(t) = \delta S_\eta(t). \tag{4.2.6}$$

by substituting these expressions into right sides of equations (4.2.5), we arrive at the following equations for current isochronal variations of coordinates $\delta\xi(t)$ and $\delta\eta(t)$:

$$\begin{aligned} \delta\xi(t) &= (1 + h_{\xi\xi})\delta S_\xi(t) + h_{\xi\eta}\delta S_\eta(t), \\ \delta\eta(t) &= h_{\eta\xi}\delta S_\xi(t) + (1 + h_{\eta\eta})\delta S_\eta(t). \end{aligned} \tag{4.2.7}$$

It is easy to be convinced that subsequent iterations of equations (4.2.5) lead accurately to the same equations for $\delta\xi(t)$ and $\delta\eta(t)$, if only in the calculations we drop terms with squares and products $h_{\xi\xi}$, $h_{\xi\eta} = h_{\eta\xi}$, and $h_{\eta\eta}$.

To search for variations of projections of the velocity of the missile $\delta u_\xi(t)$ and $\delta u_\eta(t)$, let us substitute into right sides of relations (4.1.1) expressions for $\delta\xi(t)$ and $\delta\eta(t)$, represented by equations of the same quadratic approximation (4.1.6). We obtain for the desired magnitudes these equations

$$\delta u_\xi(\sigma) = \delta V_\xi(\sigma) + \delta\xi(\sigma)\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \xi} t^2 dt + \delta\eta(\sigma)\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \eta} t^2 dt,$$
$$\delta u_\eta(\sigma) = \delta V_\eta(\sigma) + \delta\xi(\sigma)\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\eta}{\partial \xi} t^2 dt + \delta\eta(\sigma)\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\eta}{\partial \eta} t^2 dt. \quad (4.2.8)$$

Let us introduce designations

$$g_{\xi\xi}(\sigma) = \frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \xi} t^2 dt;\quad g_{\xi\eta}(\sigma) = \frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \eta} t^2 dt,$$
$$g_{\eta\xi}(\sigma) = \frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\eta}{\partial \xi} t^2 dt;\quad g_{\eta\eta}(\sigma) = \frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\eta}{\partial \eta} t^2 dt. \quad (4.2.9)$$

Then equations (4.2.8) are presented in the form

$$\delta u_\xi(\sigma) = \delta V_\xi(\sigma) + g_{\xi\xi}(\sigma)\delta\xi(\sigma) + g_{\xi\eta}(\sigma)\delta\eta(\sigma),$$
$$\delta u_\eta(\sigma) = \delta V_\eta(\sigma) + g_{\eta\xi}(\sigma)\delta\xi(\sigma) + g_{\eta\eta}(\sigma)\delta\eta(\sigma). \quad (4.2.10)$$

Similar to the previous in equations (4.2.10), instead of coefficients $g_{\xi\xi}(\sigma)$, $g_{\xi\eta}(\sigma) = g_{\eta\xi}(\sigma)$ and $g_{\eta\eta}(\sigma)$, it is possible to take their values at the calculated instant of the termination of the powered-flight section of flight of the missile σ^*. We designate these values, respectively, by $g_{\xi\xi}$, $g_{\xi\eta} = g_{\eta\xi}$ and $g_{\eta\eta}$. Let us substitute further into equations (4.2.10) expressions (4.2.5) for variations of coordinates $\delta\xi(\sigma)$ and $\delta\eta(\sigma)$ and disregard the products of coefficients g and h with arbitrary indices. Ultimately we arrive at equations for the desired isochronal variations of velocity of the missile $\delta u_\xi(\sigma)$ and $\delta u_\eta(\sigma)$ with respect to immobile system of coordinates $\xi\eta\zeta$. Changing in them the designation of argument σ by t, we obtain

$$\delta u_\xi(t) = \delta V_\xi(t) + g_{\xi\xi}\delta S_\xi(t) + g_{\xi\eta}\delta S_\eta(t),$$
$$\delta u_\eta(t) = \delta V_\eta(t) + g_{\eta\xi}\delta S_\xi(t) + g_{\eta\eta}\delta S_\eta(t). \quad (4.2.11)$$

Equations (4.2.7) and (4.2.11) are the approximate solution of the totality of differential equations (3.1.3) in a form similar for the

application in systems of inertial control of flight range of ballistic missiles.

Somewhat more bulky are equations for isochronal variations of coordinates and projections of velocity of the missile, if we take polynomials (4.1.7) for the initial approximations in equations (4.1.1) and (4.1.3). As a result, similar to the case of the quadratic approximation, it is possible to arrive at the following equations for variations of coordinates:

$$\begin{aligned}\delta\xi(t) &= (1+3h_{\xi\xi}-2j_{\xi\xi})\,\delta S_\xi(t)+(3h_{\xi\eta}-2j_{\xi\eta})\,\delta S_\eta(t)+\\ &\quad+\sigma^2(j_{\xi\xi}-h_{\xi\xi})\,\delta V_\xi(t)+\sigma^2(j_{\xi\eta}-h_{\xi\eta})\,\delta V_\eta(t),\\ \delta\eta(t) &= (3h_{\eta\xi}-2j_{\eta\xi})\,\delta S_\xi(t)+(1+3h_{\eta\eta}-2j_{\eta\eta})\,\delta S_\eta(t)+\\ &\quad+\sigma^2(j_{\eta\xi}-h_{\eta\xi})\,\delta V_\xi(t)+\sigma^2(j_{\eta\eta}-h_{\eta\eta})\,\delta V_\eta(t),\end{aligned} \tag{4.2.12}$$

and for variations of projections of the velocity, correspondingly

$$\begin{aligned}\delta u_\xi(t) &= (3g_{\xi\xi}-2i_{\xi\xi})\,\delta S_\xi(t)+(3g_{\xi\eta}-2i_{\xi\eta})\,\delta S_\eta(t)+\\ &\quad+[1+\sigma^2(i_{\xi\xi}-g_{\xi\xi})]\,\delta V_\xi(t)+\sigma^2(i_{\xi\eta}-g_{\xi\eta})\,\delta V_\eta(t),\\ \delta u_\eta(t) &= (3g_{\eta\xi}-2i_{\eta\xi})\,\delta S_\xi(t)+(3g_{\eta\eta}-2i_{\eta\eta})\,\delta S_\eta(t)+\\ &\quad+\sigma^2(i_{\eta\xi}-g_{\eta\xi})\,\delta V_\xi(t)+[1+\sigma^2(i_{\eta\eta}-g_{\eta\eta})]\,\delta V_\eta(t).\end{aligned} \tag{4.2.13}$$

In equations (4.2.12) and (4.2.13), besides designations already introduced in this section, quantities $i_{\xi\xi}$, $i_{\xi\eta}$, $i_{\eta\xi}$, $i_{\eta\eta}$ and $j_{\xi\xi}$, $j_{\xi\eta}$, $j_{\eta\xi}$, $j_{\eta\eta}$ are values of integrals

$$\begin{aligned}i_{\xi\xi}(\sigma) &= \frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \xi}\,t^3\,dt,\quad i_{\eta\eta}(\sigma)=\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\eta}{\partial \eta}\,t^3\,dt,\\ i_{\xi\eta}(\sigma) &= i_{\eta\xi}(\sigma)=\frac{1}{\sigma^3}\int_0^\sigma \frac{\partial f_\xi}{\partial \eta}\,t^3\,dt\end{aligned} \tag{4.2.14}$$

and

$$\begin{aligned}j_{\xi\xi}(\sigma) &= \frac{1}{\sigma^3}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial \xi}\,t^3\,dt^2,\quad j_{\eta\eta}(\sigma)=\frac{1}{\sigma^3}\int_0^\sigma\int_0^t \frac{\partial f_\eta}{\partial \eta}\,t^3\,dt^2,\\ h_{\xi\eta}(\sigma) &= j_{\eta\xi}(\sigma)=\frac{1}{\sigma^3}\int_0^\sigma\int_0^t \frac{\partial f_\xi}{\partial \eta}\,t^3\,dt^2,\end{aligned} \tag{4.2.15}$$

which they take with argument σ equal to σ^*, i.e., at the calculated instant of switching off of the engine.

§ 3. Simplification of Equations for Isochronal Variations of Coordinates and Projections of Velocity of the Missile at Extensions of the Powered-Flight Small in Comparison with the Radius of the Earth

Equations (4.2.7) and (4.2.10) and also (4.2.12) and (4.2.13), obtained in the previous section for variations of coordinates and projections of velocity of the missile, can without great harm for their accuracy be represented in simpler form, which does not require preliminary calculation of integrals of the type (4.2.2), (4.2.9), (4.2.14) and (4.2.15). Actually, partial derivatives of the accelerations of the force of the earth's gravity $\frac{\partial f_\xi}{\partial \xi}$, $\frac{\partial f_\xi}{\partial \eta} = \frac{\partial f_\eta}{\partial \xi}$, $\frac{\partial f_\eta}{\partial \eta}$, taking part in these integrals, are functions of moving coordinates of the calculated motion of the missile and, consequently, finally – time. However, with a small extension of the powered-flight section of the trajectory of the missile, in comparison with the radius of the earth, these functions can be substituted by their values, which refer to the point of the missile launch, and the derivatives themselves can be calculated on the assumption that the earth is a sphere with a radial distribution of density.

The last assumption, if one considers equations (2.5.4), leads to the following expressions for the mentioned derivatives of accelerations of the force of gravity:

$$\frac{\partial f_\xi}{\partial \xi} = -\frac{f_0 R^2}{\rho^3}\left(1 - 3\frac{\xi^2}{\rho^2}\right), \quad \frac{\partial f_\eta}{\partial \eta} = -\frac{f_0 R^2}{\rho^3}\left(1 - 3\frac{\eta^2}{\rho^2}\right),$$

$$\frac{\partial f_\xi}{\partial \eta} = \frac{\partial f_\eta}{\partial \xi} = 3\frac{f_0 R^2}{\rho^5}\xi\eta. \qquad (4.3.1)$$

Here ξ, η and ζ – coordinates of the arbitrary point of space and ρ – its distance to the center of the earth.

For the point of the missile launch (see § 1 of Chapter II) we have

$$\xi = \zeta = 0 \text{ and } \eta = \rho = R. \tag{4.3.2}$$

and, consequently, at this point, according to equations (4.3.1),

$$\frac{\partial f_\xi}{\partial \xi} = -\frac{f_0}{R} = -\nu^2, \quad \frac{\partial f_\xi}{\partial \eta} = \frac{\partial f_\eta}{\partial \xi} = 0,$$
$$\frac{\partial f_\eta}{\partial \eta} = \frac{2f_0}{R} = 2\nu^2, \tag{4.3.3}$$

where there is introduced designation

$$\nu = \sqrt{\frac{f_0}{R}}. \tag{4.3.4}$$

In the theory of gyroscopes and inertial navigation quantity ν, determined by equation (4.3.4), is known by the name of Schuler frequency ($\nu \approx 0.00125$ 1/s).

Using the approximate expressions (4.3.3) for derivatives $\frac{\partial f_\xi}{\partial \xi}$, $\frac{\partial f_\xi}{\partial \eta} = \frac{\partial f_\eta}{\partial \xi}$ and $\frac{\partial f_\eta}{\partial \eta}$ in equalities (4.2.2), (4.2.9), (4.2.14) and (4.2.15), we arrive at the following simple equations:

$$h_{\xi\xi}(\sigma) = -\frac{\nu^2\sigma^2}{12}, \; h_{\xi\eta}(\sigma) = h_{\eta\xi}(\sigma) = 0, \; h_{\eta\eta}(\sigma) = \frac{\nu^2\sigma^2}{6},$$
$$g_{\xi\xi}(\sigma) = -\frac{\nu^2\sigma}{3}, \quad g_{\xi\eta}(\sigma) = g_{\eta\xi}(\sigma) = 0, \quad g_{\eta\eta}(\sigma) = \frac{2\nu^2\sigma}{3}, \tag{4.3.5}$$

and also

$$f_{\xi\xi}(\sigma) = -\frac{\nu^2\sigma^2}{20}, \quad f_{\xi\eta}(\sigma) = f_{\eta\xi}(\sigma) = 0, \quad f_{\eta\eta}(\sigma) = \frac{\nu^2\sigma^2}{10},$$
$$i_{\xi\xi}(\sigma) = -\frac{\nu^2\sigma}{4}, \quad i_{\xi\eta}(\sigma) = i_{\eta\xi}(\sigma) = 0, \quad i_{\eta\eta}(\sigma) = \frac{\nu^2\sigma}{2}. \tag{4.3.6}$$

Having assumed in these equations $\sigma = \sigma^*$, one can present expressions (4.2.7) and (4.2.11) for isochronal variations of coordinates and projections of velocity of the missile with the initial quadratic approximation (4.1.6) in the following final form:

$$
\begin{aligned}
\delta\xi(t) &= \left[1 - \frac{\nu^2(\sigma^*)^2}{12}\right]\delta S_\xi(t),\\
\delta\eta(t) &= \left[1 + \frac{\nu^2(\sigma^*)^2}{6}\right]\delta S_\eta(t),\\
\delta u_\xi(t) &= \delta V_\xi(t) - \frac{\nu^2\sigma^*}{3}\,\delta S_\xi(t),\\
\delta u_\eta(t) &= \delta V_\eta(t) + \frac{2\nu^2\sigma^*}{3}\,\delta S_\eta(t).
\end{aligned}
\qquad (4.3.7)
$$

Similarly, for the case of the initial approximation of functions $\delta\xi(t)$ and $\delta\eta(t)$ in the form of polynomials (4.1.7), on the basis of equations (4.2.12) and (4.2.13), we obtain

$$
\begin{aligned}
\delta\xi(t) &= \left[1 - \frac{3\nu^2(\sigma^*)^2}{20}\right]\delta S_\xi(t) + \frac{\nu^2(\sigma^*)^2}{30}\,\delta V_\xi(t),\\
\delta\eta(t) &= \left[1 + \frac{3}{10}\nu^2(\sigma^*)^2\right]\delta S_\eta(t) - \frac{1}{15}\nu^2(\sigma^*)^2\,\delta V_\eta(t),\\
\delta u_\xi(t) &= \left[1 + \frac{\nu^2(\sigma^*)^2}{12}\right]\delta V_\xi(t) - \frac{\nu^2\sigma^*}{2}\,\delta S_\xi(t),\\
\delta u_\eta(t) &= \left[1 - \frac{\nu^2(\sigma^*)^2}{6}\right]\delta V_\eta(t) + \nu^2\sigma^*\,\delta S_\eta(t).
\end{aligned}
\qquad (4.3.8)
$$

Equations (4.3.7) or (4.3.8), which approximately consider the change in action on the missile of the force of the earth's gravity (because of the motion of the missile not according to the assigned calculated law), should be more accurate than similar equations (3.2.2) and (3.2.6) given in Chapter III.

§ 4. Construction of Ballistic Equations in Which the Effect of Changes in Acceleration of the Force of the Earth's Gravity Is Approximately Taken Account

In the previous paragraph equations (4.3.7) and also (4.3.8) were obtained for the determination of variations of coordinates and projections of the velocity of the missile with the approximate calculation of the effect of the change in the force of the earth's gravity with motion according to the law which is somewhat distinguished from the calculated. The mentioned equations allow more accurately, in comparison with expressions (3.3.13) and (3.4.16)

obtained in Chapter III, constructing the left side of the basic ballistic equation (2.7.6) aboard the missile, using readings of the same integrators of accelerations. For this we will use in the beginning equations (4.3.7) and substitute into equation (2.7.6) expressions for $\delta\xi(t)$, $\delta\eta(t)$, $\delta u_\xi(t)$ and $\delta u_\eta(t)$ determined by them; we obtain

$$\begin{aligned}\delta S_\xi(t)\left\{\left[1-\frac{\nu^2(\sigma^*)^2}{12}\right]\frac{\partial l}{\partial \xi}-\frac{\nu^2\sigma^*}{3}\frac{\partial l}{\partial u_\xi}\right\}&+\\ +\delta S_\eta(t)\left\{\left[1+\frac{\nu^2(\sigma^*)^2}{6}\right]\frac{\partial l}{\partial \eta}+\frac{2\nu^2\sigma^*}{3}\frac{\partial l}{\partial \eta}\right\}&+\\ +\delta V_\xi(t)\frac{\partial l}{\partial u_\xi}+\delta V_\eta(t)\frac{\partial l}{\partial u_\eta}&=\\ =-(t-\sigma^*)\left[a_\xi^*(\sigma^*)\frac{\partial l}{\partial u_\xi}+a_\eta^*(\sigma^*)\frac{\partial l}{\partial u_\eta}\right]&. \qquad (4.4.1)\end{aligned}$$

Further, similar to that which was done in § 2 of Chapter III, let us present variations of projections of velocity $\delta V_\xi(t)$ and $\delta V_\eta(t)$ in the form of differences (3.2.3) and use equations (3.2.12). In this case just as in the conversion of equation (3.2.10), terms standing in the right side of equality (4.4.1) are reduced with similar terms of its left side. Taking into account still equations (3.2.7), we arrive at the equation

$$\begin{aligned}[S_\xi(t)-S_\xi^*(t)]\,A_\xi'+[S_\eta(t)-S_\eta^*(t)]A_\eta'+V_\xi(t)\frac{\partial l}{\partial u_\xi}&+\\ +V_\eta(t)\frac{\partial l}{\partial u_\eta}=V_\xi^*(\sigma^*)\frac{\partial l}{\partial u_\xi}+V_\eta^*(\sigma^*)\frac{\partial l}{\partial u_\eta}&, \qquad (4.4.2)\end{aligned}$$

in which

$$\begin{aligned}A_\xi'&=\frac{\partial l}{\partial \xi}-\frac{\nu^2(\sigma^*)^2}{12}\frac{\partial l}{\partial \xi}-\frac{\nu^2\sigma^*}{3}\frac{\partial l}{\partial u_\xi},\\ A_\eta'&=\frac{\partial l}{\partial \eta}+\frac{\nu^2(\sigma^*)^2}{6}\frac{\partial l}{\partial \eta}+\frac{2\nu^2\sigma^*}{3}\frac{\partial l}{\partial u_\eta}.\end{aligned} \qquad (4.4.3)$$

Equation (4.4.2) differs from equation (3.2.13) only by coefficients of isochronal variations of projections of the apparent path $\delta S_\xi(t)$ and $\delta S_\eta(t)$. Such coefficients in equation (3.2.13) are, respectively, partial derivatives $\frac{\partial l}{\partial \xi}$ and $\frac{\partial l}{\partial \eta}$ and in equation (4.4.2) –

are quantities A'_ξ and A'_η distinguished little from them. Therefore, subsequent conversions of equat on (4.4.2) can be produced exactly as in § 3 of the previous chapter. Then instead of equation (3.3.13), we will arrive at the following:

$$V_\lambda(t) + p'[S_{\mu'}(t) - S^*_{\mu'}(t)] = V^*_\lambda(\sigma^*). \qquad (4.4.4)$$

The left sides of equations (4.4.2) and (4.4.4) are constructed aboard the missile with the help of the same technical means as for equations (3.2.13) and (3.3.13), i.e., two integrators of accelerations and the appropriate computer, which includes elements of repeated integration. In this case the axis of sensitivity of one of the integrators is at the same constant angle λ to axis ξ as in the construction aboard the missile of the left side of equation (3.3.13). As regards the axis of sensitivity of the other integrator, the latter should be inclined toward the axis ξ at an angle of μ', which is little distinguished from the corresponding angle μ. Similar to equality (3.3.8) the magnitude of angle μ' is found from relations

$$A'_\xi = N'\cos\mu', \qquad A'_\eta = N'\sin\mu', \qquad (4.4.5)$$

where, of course,

$$N' = \sqrt{(A'_\xi)^2 + (A'_\eta)^2}. \qquad (4.4.6)$$

In turn the coefficient p' is distinguished little from coefficient p of equation (3.3.13) and is determined by equation

$$p' = \frac{N'}{M}. \qquad (4.4.7)$$

Here quantity M is the same as that in equation (3.3.4).

Possible also is the conversion of the ballistic equation (4.4.4) to the form

$$\int_0^t K'(t)\, a_{x'}(t)\, dt = C'. \qquad (4.4.8)$$

similar to the form of equation (3.4.16). The use of equation (4.4.8) allows solving the problem about inertial control of flight range of ballistic missiles just as in § 4 of Chapter III by means of one meter of acceleration with an alternating direction of the axis of sensitivity. In equation (4.4.8) the variable coefficient $K'(t)$ is formed by the same equations (3.4.11) as coefficient $K(t)$ in equation (3.4.16) but with the replacement of angle $\kappa(t)$ by angle $\kappa'(t)$ and derivatives $\frac{\partial l}{\partial \xi}$ and $\frac{\partial l}{\partial \eta}$, respectively, by quantities A'_{ξ} and A'_{η}.

The use in the transformation of the basic ballistic equation (2.7.6) of more complex equations (4.3.8) for coordinates and projections of velocity of the missile leads, naturally, to more bulky results in comparison with equations given in the beginning of this section. Instead of equation (4.4.1), in this case we arrive at the more complex equation

$$\begin{aligned} \delta S_{\xi}(t) & \left\{\left[1-\frac{3\nu^2(\sigma^*)^2}{20}\right]\frac{\partial l}{\partial \xi}-\frac{\nu^2(\sigma^*)^2}{2}\frac{\partial l}{\partial u_{\xi}}\right\}+ \\ & +\delta S_{\eta}\left\{\left[1+\frac{3}{10}\nu^2(\sigma^*)^2\right]\frac{\partial l}{\partial \eta}+\nu^2\sigma^*\frac{\partial l}{\partial u_{\eta}}\right\}+ \\ & +\delta V_{\xi}\left\{\frac{\nu^2(\sigma^*)^3}{30}\frac{\partial l}{\partial \xi}+\left[1+\frac{\nu^2(\sigma^*)^2}{12}\right]\frac{\partial l}{\partial u_{\xi}}\right\}+ \\ & +\delta V_{\eta}\left\{-\frac{\nu^2(\sigma^*)^3}{15}\frac{\partial l}{\partial \eta}+\left[1-\frac{\nu^2(\sigma^*)^2}{6}\right]\frac{\partial l}{\partial u_{\eta}}\right\}= \\ & =-(t-\sigma^*)\left[a_{\xi}^*(\sigma^*)\frac{\partial l}{\partial u_{\xi}}+a_{\eta}^*(\sigma^*)\frac{\partial l}{\partial u_{\eta}}\right]. \end{aligned} \quad (4.4.9)$$

In turn equation (4.4.2) is replaced by the following:

$$\begin{aligned} & [S_{\xi}(t)-S_{\xi}^*(t)]A_{\xi}^*+[S_{\eta}(t)-S_{\eta}^*(t)]A_{\eta}^*+V_{\xi}(t)B_{\xi}^*+ \\ & +V_{\eta}(t)B_{\eta}^*-V_{\xi}^*(t)\left(B_{\xi}^*-\frac{\partial l}{\partial u_{\xi}}\right)-V_{\eta}^*(t)\left(B_{\eta}^*-\frac{\partial l}{\partial u_{\eta}}\right)= \\ & =V_{\xi}^*(\sigma^*)\frac{\partial l}{\partial u_{\xi}}+V_{\eta}^*(\sigma^*)\frac{\partial l}{\partial u_{\eta}}. \end{aligned} \quad (4.4.10)$$

In it these designations are introduced

$$A_{\xi}^*=\left[1-\frac{3\nu^2(\sigma^*)^2}{20}\right]\frac{\partial l}{\partial \xi}-\frac{\nu^2\sigma^*}{2}\frac{\partial l}{\partial u_{\xi}}.$$

$$A_\eta'' = \left[1 + \frac{3\nu^2(\sigma'')^2}{10}\right]\frac{\partial l}{\partial \eta} + \nu^2\sigma''\frac{\partial l}{\partial u_\eta},$$
$$B_\xi'' = \frac{\nu^2(\sigma'')^2}{30}\frac{\partial l}{\partial \xi} + \left[1 + \frac{\nu^2(\sigma'')^2}{12}\right]\frac{\partial l}{\partial u_\xi}, \qquad (4.4.11)$$
$$B_\eta'' = -\frac{\nu^2(\sigma'')^2}{15}\frac{\partial l}{\partial \eta} + \left[1 - \frac{\nu^2(\sigma'')^2}{6}\right]\frac{\partial l}{\partial u_\eta}.$$

Similarly, equation (4.4.4), when using equations (4.3.8), is replaced by the following:

$$V_{\lambda''}(t) + p''[S_{\mu''}(t) - S_{\mu''}^{\bullet}(t)] = V_{\lambda''}^{\bullet}(\sigma'') +$$
$$+ \frac{t-\sigma''}{M''}\left[\left(B_\xi'' - \frac{\partial l}{\partial u_\xi}\right)a_\xi^{\bullet}(\sigma'') + \left(B_\eta'' - \frac{\partial l}{\partial u_\eta}\right)a_\eta^{\bullet}(\sigma'')\right]. \qquad (4.4.12)$$

Here λ" and μ" are directions of axes of sensitivity of two integrators of accelerations, which, respectively, differ little from λ- and μ-directions, introduced in § 3 of Chapter III. Angles λ" and μ", which these directions from with axis ξ, are determined by relation

$$A_\xi'' = N''\cos\mu'', \quad A_\eta'' = N''\sin\mu'',$$
$$B_\xi'' = M''\cos\lambda'', \quad B_\eta'' = M''\sin\lambda''. \qquad (4.4.13)$$

Coefficient p'' in equation (4.4.12) is expressed by equation

$$p'' = \frac{N''}{M''} = \sqrt{\frac{(A_\xi'')^2 + (A_\eta'')^2}{(B_\xi'')^2 + (B_\eta'')^2}}. \qquad (4.4.14)$$

The construction of the right side of the ballistic equation in the form (4.4.12) aboard the missile is complicated, as compared to the case of the equation of the form (4.4.4), by the necessity of introduction into the computer of an additional term linearly dependent on time. Finally, subsequent conversions of equation (4.4.12), specifically, to the form similar to the ballistic equation (4.4.8) and also (3.5.2) are allowed.

In conclusion let us note that the selection for the specific system of control of the flight range of one of the ballistic

equations given in this and previous chapters should be produced on the basis of estimates of quantities of the so-called systematic errors inherent to the system of inertial control of the given type of rocket. Systematic errors are understood usually as those errors in the determining of flight range of the missile, which appear exclusively because of simplifications made in the derivation of the appropriate ballistic equation. Because of this, even with the accurate switching off of the engine of the rocket at the instant indicated by the ballistic equation and with the accurate operation of all remaining instruments and devices of the system of inertial control, the actual flight range of the missile can be somewhat distinguished from that assigned according to the calculation. As regards to the technical difficulties of the construction of systems of inertial control of flight range of ballistic missiles, they are included in the manufacture of meters and integrators of the apparent accelerations (i.e., newtonmeters and impulse meters) with extremely small instrumental errors and gyroscopic instruments with minimum servicing of their stabilized axes relative to directions on fixed star. The same refers to the immediate switching off of the engine on the signal of the achievement by the ballistic function of a value corresponding to the assigned flight range of the missile.

CHAPTER V

CONTROL OF LATERAL MOTION OF THE BALLISTIC MISSILE

§ 1. Lateral Deviation of the Missile from the Target, Expressed in the Form of a Function of Changes in Parameters of the End of the Powered-Flight Section in the Starting System of Coordinates

In the solution of the problem about the elimination of lateral deviation in the missile from the assigned target, there appear additional difficulties connected with the fact that of the duration of the free-flight section of real motion of the missile, as a rule, is somewhat distinguished from its calculated value. In view of this, control according to the earlier assigned law of the motion of the missile in a direction perpendicular to the programmed plane, i.e., plane xy of the starting system of coordinates xyz or (in another possible variant) to plane $\xi\eta$ of the nonrotating system $\xi\eta\zeta$, in general does not provide the absence of lateral deviation in the missile from the target. Really, because of the laws of mechanics the missile on the greater part of the free-flight section of its flight is moved with minute deviations from a certain plane, which does not change its orientation relative to directions at fixed stars (the mentioned deviations are connected basically with the nonsphericity of the earth). Therefore, if the duration of the free-flight section is not equal to its calculated value, then because of the rotation of the earth lateral deviation in the missile from the target[1] continuously appears. However, with

[1]An exception is the case of flight of a missile in the plane of the equator and from one pole on another.

small distinction in the actual motion of the missile from calculated this deviation should also be small. Correspondingly the control of motion of the missile in the powered-flight section in a lateral direction can be reduced only to the requirement of the fulfillment of an earlier assigned (calculated) law of motion in the projection on the z axis (or in another variant - on axis ζ). However, for a more accurate elimination of lateral deviation in the missile from the target it is necessary to take into account how at current instant of time its real motion in a longitudinal direction is distinguished from the calculated. For this purpose one can use current readings of instruments of the control of flight range of the missile.

It is obvious that an especially accurate control of lateral motion of the missile is necessary only during a short interval of time when cutoff of its engine can occur in flight at the assigned range. At the remaining time of the powered-flight section the accuracy of control of lateral motion can be somewhat less. The duration of the mentioned interval of allowed cutoffs of the engine depend on the quality of control of motion of the missile according to the assigned program.

In the starting system of coordinates xyz (see § 1 of the second chapter, Fig. 9) quantity b - of lateral deviation in the missile from the target, just as the range L, is a function of coordinates x, y, z and projections v_x, v_y, v_z of its velocity relative to this system of coordinates at the instant of termination of the powered-flight section of the flight. Thus,

$$b = b(x, y, z, v_x, v_y, v_z). \tag{5.1.1}$$

The astronomical time, which corresponds to instant of termination of the powered-flight section, does not play any role here as in equation (2.1.2), unless one does not take into account the negligible effect of such factors as the mutual arrangement of the earth and moon or the earth and sun.

With the replacement of arguments of function (5.1.1) by their calculated values x^*, y^*, z^*, v_x^*, v_y^*, and v_z^* it should turn into zero, since the lateral deviation in this case is absent. Thus,

$$b(x^*, y^*, z^*, v_x^*, v_y^*, v_z^*) = 0 \tag{5.1.2}$$

and, consequently,

$$\Delta b = b(x, y, z, v_x, v_y, v_z) - b(x^*, y^*, z^*, v_x^*, v_y^*, v_z^*) = b, \tag{5.1.3}$$

i.e., the error in the lateral deviation Δb is equal to the lateral deviation b.

In accordance with the Taylor series for the function of many variables, we have, correct to smalls of the first order inclusively, the following expansion:

$$\begin{aligned} b = (x - x^*)\frac{\partial b}{\partial x} + (y - y^*)\frac{\partial b}{\partial y} + (z - z^*)\frac{\partial b}{\partial z} + \\ + (v_x - v_x^*)\frac{\partial b}{\partial v_x} + (v_y - v_y^*)\frac{\partial b}{\partial v_y} + (v_z - v_z^*)\frac{\partial b}{\partial v_z}. \end{aligned} \tag{5.1.4}$$

Here derivatives $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$, $\frac{\partial b}{\partial z}$, $\frac{\partial b}{\partial v_x}$, $\frac{\partial b}{\partial v_y}$ and $\frac{\partial b}{\partial v_z}$ are taken at calculated values of their arguments and, consequently, for each calculated case of flight of the missile are earlier known quantities. They can be determined in a way similar to the ballistic coefficients given in § 1 of the second chapter.

§ 2. Lateral Deviation as a Function of Parameters of the End of the Powered-Flight Section in the Nonrotating System of Coordinates

In examining the problems of inertial control of lateral motion of the ballistic missile, the starting system of coordinates is inconvenient. Therefore, let us introduce the same nonrotating system of coordinates $\xi\eta\zeta$ as that in the examining of the theory of inertial control of range (see § 2 of the second chapter). Current coordinates $x(t)$, $y(t)$, $z(t)$ and projections $v_x(t)$, $v_y(t)$ and $v_z(t)$ of the velocity of the missile relative to the starting system of

coordinates xyz are expressed in this case by current coordinates $\xi(t)$, $\eta(t)$, $\zeta(t)$ and projections $u_\xi(t)$, $u_\eta(t)$ and $u_\zeta(t)$ of the velocity of the missile in the nonrotating system of coordinates $\xi\eta\zeta$ by equations (2.2.5) and (2.3.5). Entering into the mentioned equations also in evident form (through angle $\phi = Ut$) is time t, which passed at the instant of launch of the missile. In view of this, in accordance with equation (5.1.1), the magnitude of lateral deviation in the missile from the target can be represented in the form of a function[1]

$$b = b(\xi, \eta, \zeta, u_\xi, u_\eta, u_\zeta; \sigma) \tag{5.2.1}$$

of coordinates ξ, η, ζ and projections u_ξ, u_η, u_ζ of the velocity of the missile relative to the system of coordinates $\xi\eta\zeta$ at the instant $t = \sigma$ of termination of the powered-flight section and also its very duration σ.

Calculated values of coordinates x^*, y^*, z^* and projections v_x^*, v_y^*, v_z^* of velocity of the missile relative to the starting system xyz correspond to certain calculated values of coordinates ξ^*, η^*, ζ^* and projections u_ξ^*, u_η^*, u_ζ^* of velocity of the missile in the nonrotating system $\xi\eta\zeta$. It is obvious that function (5.2.1), which represents the lateral deviation of the missile from the target, will turn into zero at these values of their arguments if, furthermore, we assume the duration of the powered-flight section is also equal to its calculated value σ^*. Thus,

$$b(\xi^*, \eta^*, \zeta^*, u_\xi^*, u_\eta^*, u_\zeta^*; \sigma^*) = 0. \tag{5.2.2}$$

Expanding function (5.2.1) in Taylor series and being limited in the latter only by terms of the first order of smallness, we obtain, taking into account equality (5.2.2), the equation

$$b = (\xi - \xi^*)\frac{\partial b}{\partial \xi} + (\eta - \eta^*)\frac{\partial b}{\partial \eta} + (\zeta - \zeta^*)\frac{\partial b}{\partial \zeta} + \\ + (u_\xi - u_\xi^*)\frac{\partial b}{\partial u_\xi} + (u_\eta - u_\eta^*)\frac{\partial b}{\partial u_\eta} + (u_\zeta - u_\zeta^*)\frac{\partial b}{\partial u_\zeta} + (\sigma - \sigma^*)\frac{\partial b}{\partial \sigma}. \tag{5.2.3}$$

---

[1]Here, just as in § 4 of the second chapter, designations (2.4.1) are accepted.

in which partial derivatives $\frac{\partial b}{\partial \xi}$, $\frac{\partial b}{\partial \eta}$, $\frac{\partial b}{\partial \zeta}$, $\frac{\partial b}{\partial u_\xi}$, $\frac{\partial b}{\partial u_\eta}$, $\frac{\partial b}{\partial u_\zeta}$ and $\frac{\partial b}{\partial \sigma}$ are constant coefficients, since their arguments ξ, η, ζ, u_ξ, u_η, u_ζ and σ are taken equal to their calculated values, i.e., respectively ξ^*, η^*, ζ^*, u_ξ^*, u_η^*, u_ζ^* and σ^*. Magnitudes of these coefficients depend finally on the selection of a certain calculated case of flight of the missile. In a way similar to the appropriate partial derivative of function l, which represents the flight range of the missile, they are expressed by equations of the type (2.4.5) and (2.4.6) in terms of partial derivatives $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$, $\frac{\partial b}{\partial z}$, $\frac{\partial b}{\partial v_x}$, $\frac{\partial b}{\partial v_y}$, $\frac{\partial b}{\partial v_z}$, and also quantities x^*, y^*, z^*, v_x^*, v_y^*, v_z^*.

Equations (5.2.3), correct to squares and products of differences $\xi-\xi^*$, $\eta-\eta^*$, $\zeta-\zeta^*$, $u_\xi-u_\xi^*$, $u_\eta-u_\eta^*$, $u_\zeta-u_\zeta^*$, $\sigma-\sigma^*$, determines the lateral deviation in the missile from the target when magnitudes of coordinates and projections of velocity of the missile at the end of the powered-flight section are not equal to their calculated values.

With real motion of the missile neither the instant of switching off of the engine σ nor its coordinates ξ, η and projection of velocity u_ξ, u_η, which correspond to this instant of time, are known earlier. Therefore, for elimination with the above-mentioned accuracy of lateral deviation in the missile from the target, a continuous change is necessary in quantities $\zeta(t)$ and $u_\zeta(t)$ characterizing its motion in a lateral direction during a certain time interval of the end of the calculated powered-flight section. The purpose of such a control is the continuous reduction to zero of function

$$\begin{aligned}\beta(t) = [\xi(t)-\xi^*]\frac{\partial b}{\partial \xi} + [\eta(t)-\eta^*]\frac{\partial b}{\partial \eta} + [\zeta(t)-\zeta^*]\frac{\partial b}{\partial \zeta} + \\ + [u_\xi(t)-u_\xi^*]\frac{\partial b}{\partial u_\xi} + [u_\eta(t)-u_\eta^*]\frac{\partial b}{\partial u_\eta} + \\ + [u_\zeta(t)-u_\zeta^*]\frac{\partial b}{\partial u_\zeta} + (t-\sigma^*)\frac{\partial b}{\partial \sigma}\end{aligned} \tag{5.2.4}$$

in the whole interval of possible instants of the switching off of the engine for a missile of the given system. Actually, function

$\beta(t)$, according to equations (2.4.1) and (5.2.3), turns into the magnitude of lateral deviation in the missile b if we assume in it argument t equal to the actual value of the instant of the switching off of the engine σ.

The current value of function $\beta(t)$, which we will call the *function of lateral deviation*, can be used as the controlling signal in the system of control of lateral motion of the missile. This system should operate in such a way that function $\beta(t)$ would turn into zero because of the corresponding change in current magnitudes of the coordinate of the missile $\zeta(t)$ and its time derivative $u_\zeta(t)$.

§ 3. Construction of Corresponding Terms of the Function of Lateral Deviation Dependent Only on the Basic Motion of the Missile

As was shown in the previous section, for the inertial control of lateral motion it is necessary to reproduce the current value of function $\beta(t)$ aboard the missile. The totality of terms (5.2.4) making up this function is decomposed into two groups $\beta_1(t)$ and $\beta_2(t)$. The first of them is the sum

$$\beta_1(t) = [\xi(t) - \xi^*]\frac{\partial b}{\partial \xi} + [\eta(t) - \eta^*]\frac{\partial b}{\partial \eta} + [u_\xi(t) - u_\xi^*]\frac{\partial b}{\partial u_\xi} + [u_\eta(t) - u_\eta^*]\frac{\partial b}{\partial u_\eta} + (t - \sigma^*)\frac{\partial b}{\partial \sigma}, \qquad (5.3.1)$$

the magnitude of which is determined by the basic motion of the missile, i.e., by the motion of its projection on plane $\xi\eta$ of the nonrotating system of coordinates $\xi\eta\zeta$. The second group

$$\beta_2(t) = [\zeta(t) - \zeta^*]\frac{\partial b}{\partial \zeta} + [u_\zeta(t) - u_\zeta^*]\frac{\partial b}{\partial u_\zeta} \qquad (5.3.2)$$

consists of two terms containing, respectively, quantities $\zeta(t)$ and $u_\zeta(t)$.

Some methods of the construction of the current value of function $\beta_1(t)$ aboard the missile are examined in the following

section. As regards the sum (5.3.1), then it, according to its structure, is completely similar to equation (2.4.9) for the initial ballistic function $\varepsilon(t)$. Therefore, the approximate construction of the sum (5.3.1) can be produced aboard the missile by the same methods as were used for the construction of function $\varepsilon(t)$. Let us examine, for example, the construction of this sum aboard the missile by means of using readings of two integrators of accelerations, the axes of sensitivity of which are oriented respectively parallel to axes ξ and η of the nonrotating system of coordinates $\xi\eta\zeta$. Similar to equation (2.7.4), the sum (5.3.1) can be represented in the form

$$\begin{aligned}\beta_1(t) = {} & [\xi(t)-\xi^*(t)]\frac{\partial b}{\partial \xi}+[\eta(t)-\eta^*(t)]\frac{\partial b}{\partial \eta}+ \\ & +[u_\xi(t)-u_\xi^*(t)]\frac{\partial b}{\partial u_\xi}+[u_\eta(t)-u_\eta^*(t)]\frac{\partial b}{\partial u_\eta}+ \\ & +[\xi^*(t)-\xi^*]\frac{\partial b}{\partial \xi}+[\eta^*(t)-\eta^*]\frac{\partial b}{\partial \eta}+[u_\xi^*(t)-u_\xi^*]\frac{\partial b}{\partial u_\xi}+ \\ & +[u_\eta^*(t)-u_\eta^*]\frac{\partial b}{\partial u_\eta}+(t-\sigma^*)\frac{\partial b}{\partial \sigma}.\end{aligned} \tag{5.3.3}$$

With subsequent conversion of the sum $\beta_1(t)$ we will use equations (2.7.1) and (2.7.2) for isochronal variations of coordinates and projections of the velocity of the missile and also equations of the expansion in Taylor series (being limited only to terms of the first order of smallness)

$$\begin{aligned}\xi^*(t)-\xi^* &= \xi^*(t)-\xi^*(\sigma^*)=(t-\sigma^*)\frac{d\xi^*(\sigma^*)}{dt}, \\ \eta^*(t)-\eta^* &= \eta^*(t)-\eta^*(\sigma^*)=(t-\sigma^*)\frac{d\eta^*(\sigma^*)}{dt},\end{aligned} \tag{5.3.4}$$

and in exactly the same manner

$$\begin{aligned}u_\xi^*(t)-u_\xi^* &= u_\xi^*(t)-u_\xi^*(\sigma^*)=(t-\sigma^*)\frac{du_\xi^*(\sigma^*)}{dt}, \\ u_\eta^*(t)-u_\eta^* &= u_\eta^*(t)-u_\eta^*(\sigma^*)=(t-\sigma^*)\frac{du_\eta^*(\sigma^*)}{dt}.\end{aligned} \tag{5.3.5}$$

Taking into account, furthermore, that in accordance with (2.6.3)

$$\frac{du_\xi^*(\sigma^*)}{dt} = \frac{d^2\xi^*(\sigma^*)}{dt^2}, \qquad \frac{du_\eta^*(\sigma^*)}{dt} = \frac{d^2\eta^*(\sigma^*)}{dt^2}, \tag{5.3.6}$$

we transform the expression for sum $\beta_1(t)$ to the form

$$\beta_1(t) = \delta\xi(t)\frac{\partial b}{\partial \xi} + \delta\eta(t)\frac{\partial b}{\partial \eta} + \delta u_\xi(t)\frac{\partial b}{\partial u_\xi} + \delta u_\eta(t)\frac{\partial b}{\partial u_\eta} + (t-\sigma^*)\left[\frac{d\xi^*(\sigma^*)}{dt}\frac{\partial b}{\partial \xi} + \frac{d\eta^*(\sigma^*)}{dt}\frac{\partial b}{\partial \eta} + \frac{d^2\xi^*(\sigma^*)}{dt^2}\frac{\partial b}{\partial u_\xi} + \frac{d^2\eta^*(\sigma^*)}{dt^2}\frac{\partial b}{\partial u_\eta} + \frac{\partial b}{\partial \sigma}\right]. \tag{5.3.7}$$

Just as in the third chapter, let us disregard in the beginning the effect of the change in the acceleration of the force of gravity during motion of the rocket not according to the calculated law. In accordance with equations (3.2.6), (3.2.7), (3.2.2) and (3.2.3), we obtain at small values of the difference $t - \sigma^*$ relations

$$\begin{aligned} \delta\xi(t) &= \delta S_\xi(t) = S_\xi(t) - S_\xi^*(t) = S_\xi(t) - [S_\xi^*(\sigma^*) + (t-\sigma^*)V_\xi^*(\sigma^*)], \\ \delta\eta(t) &= \delta S_\eta(t) = S_\eta(t) - S_\eta^*(t) = S_\eta(t) - [S_\eta^*(\sigma^*) + (t-\sigma^*)V_\eta^*(\sigma^*)], \end{aligned} \tag{5.3.8}$$

and also

$$\begin{aligned} \delta u_\xi(t) &= \delta V_\xi(t) = V_\xi(t) - V_\xi^*(t) = V_\xi(t) - [V_\xi^*(\sigma^*) + (t-\sigma^*)a_\xi^*(\sigma^*)], \\ \delta u_\eta(t) &= \delta V_\eta(t) = V_\eta(t) - V_\eta^*(t) = V_\eta(t) - [V_\eta^*(\sigma^*) + (t-\sigma^*)a_\eta^*(\sigma^*)]. \end{aligned} \tag{5.3.9}$$

Used here again are equations of the expansion in Taylor series, but for projections $S_\xi^*(t)$, $S_\eta^*(t)$ of the apparent path of the missile and projections $V_\xi^*(t)$, $V_\eta^*(t)$ of its apparent velocity in the calculated motion equations (3.2.5) and (3.2.9) are taken into account.

By means of equalities (5.3.8) and (5.3.9) the sum (5.3.7) is reduced to the form

$$\begin{aligned}\beta_1(t) = {} & S_\xi(t)\frac{\partial b}{\partial \xi} + S_\eta(t)\frac{\partial b}{\partial \eta} + V_\xi(t)\frac{\partial b}{\partial u_\xi} + V_\eta(t)\frac{\partial b}{\partial u_\eta} - \\ & -\left[S_\xi^*(\sigma^*)\frac{\partial b}{\partial \xi} + S_\eta^*(\sigma^*)\frac{\partial b}{\partial \eta} + V_\xi^*(\sigma^*)\frac{\partial b}{\partial u_\xi} + V_\eta^*(\sigma^*)\frac{\partial b}{\partial u_\eta}\right] + \\ & + (t-\sigma^*)\left\{\left[\frac{d\xi^*(\sigma^*)}{dt} - V_\xi^*(\sigma^*)\right]\frac{\partial b}{\partial \xi} + \left[\frac{d\eta^*(\sigma^*)}{dt} - V_\eta^*(\sigma^*)\right]\frac{\partial b}{\partial \eta} + \right. \\ & \left. + \left[\frac{d^2\xi^*(\sigma^*)}{dt^2} - a_\xi^*(\sigma^*)\right]\frac{\partial b}{\partial u_\xi} + \left[\frac{d^2\eta^*(\sigma^*)}{dt^2} - a_\eta^*(\sigma^*)\right]\frac{\partial b}{\partial u_\eta} + \frac{\partial b}{\partial \sigma}\right\}. \end{aligned} \qquad (5.3.10)$$

In the last equation we can produce further simplifications, since, according to equality (2.6.1), we have

$$\begin{aligned}\frac{d^2\xi^*(t)}{dt^2} - a_\xi^*(t) &= f_\xi[\xi^*(t), \eta^*(t); t], \\ \frac{d^2\eta^*(t)}{dt^2} - a_\eta^*(t) &= f_\eta[\xi^*(t), \eta^*(t); t] \end{aligned} \qquad (5.3.11)$$

as a consequence, as a result of the integration with respect to time within limits of $t = 0$ to $t = \sigma^*$, allowing for equations (3.2.5)

$$\begin{aligned}\frac{d\xi^*(\sigma^*)}{dt} - V_\xi^*(\sigma^*) &= \int_0^{\sigma^*} f_\xi[\xi^*(t), \eta^*(t); t]\,dt + \frac{d\xi^*(0)}{dt}, \\ \frac{d\eta^*(\sigma^*)}{dt} - V_\eta^*(\sigma^*) &= \int_0^{\sigma^*} f_\eta[\xi^*(t), \eta^*(t); t]\,dt + \frac{d\eta^*(0)}{dt}. \end{aligned} \qquad (5.3.12)$$

Equations (5.3.11) and (5.3.12) can be used in the subsequent conversion of expression (5.3.10) for the sum $\beta_1(t)$. As a result let us obtain the following final expression for this sum:

$$\begin{aligned}\beta_1 = {} & S_\xi(t)\frac{\partial b}{\partial \xi} + S_\eta(t)\frac{\partial b}{\partial \eta} + V_\xi(t)\frac{\partial b}{\partial u_\xi} + V_\eta(t)\frac{\partial b}{\partial u_\eta} - \\ & - C + (t-\sigma^*)D. \end{aligned} \qquad (5.3.13)$$

Introduced here are designations

$$C = S_\xi^*(\sigma^*)\frac{\partial b}{\partial \xi} + S_\eta^*(\sigma^*)\frac{\partial b}{\partial \eta} + V_\xi^*(\sigma^*)\frac{\partial b}{\partial u_\xi} + V_\eta^*(\sigma^*)\frac{\partial b}{\partial u_\eta}, \qquad (5.3.14)$$

$$D=\frac{\partial b}{\partial \xi}\int_{0}^{s^*} f_\xi[\xi^*(t),\eta^*(t);t]\,dt+\frac{\partial b}{\partial \eta}\int_{0}^{s^*} f_\eta[\xi^*(t),\eta^*(t);t]\,dt+$$
$$+f_\xi[\xi^*(s^*),\eta^*(s^*);s^*]\frac{\partial b}{\partial u_\xi}+f_\eta[\xi^*(s^*),\eta^*(s^*);s^*]\frac{\partial b}{\partial u_\eta}+$$
$$+\left[\frac{\partial b}{\partial s}+\frac{\partial b}{\partial \xi}\frac{d\xi^*(0)}{dt}+\frac{\partial b}{\partial \eta}\frac{d\eta^*(0)}{dt}\right]. \tag{5.3.15}$$

It is obvious that C and D are constant quantities, which can be determined earlier for each specific case of flight of the missile.

For the construction of expression (5.3.13) aboard the missile, one can use the same integrators of accelerations and elements of repeated integration as in the solution of the problem on inertial control of flight range by the method given in § 2 of the third chapter. Furthermore, an additional computer and clock mechanism are necessary here.

If, however, the control of flight range of the missile is produced with the help of integrators whose axes of sensitivity are parallel to λ- and μ-directions, then for the formation of expression (5.3.13) there will be required, furthermore, an auxiliary device of the type of converter of coordinates, which continuously solves the system of equations (see § 3 of the third chapter)

$$\begin{aligned} V_\xi(t)\cos\lambda+V_\eta(t)\sin\lambda&=V_\lambda(t),\\ V_\xi(t)\cos\mu+V_\eta(t)\sin\mu&=V_\mu(t) \end{aligned} \tag{5.3.16}$$

relative to quantities $V_\xi(t)$ and $V_\eta(t)$ according to data of current readings $V_\lambda(t)$ and $V_\mu(t)$ above-mentioned integrators.

§ 4. Construction of Main Terms of the Function of Lateral Deviation

The second group of corresponding terms of equation (5.2.4) for function $\beta(t)$, as was indicated in the previous section, is the sum of two terms

$$[\zeta(t)-\zeta^*]\frac{\partial b}{\partial \zeta} \quad \text{and} \quad [v_\zeta(t)-u_\zeta^*]\frac{\partial b}{\partial u_\zeta}. \tag{5.4.1}$$

They can be called the main terms of the function of lateral deviation $\beta(t)$, since they play the main role in the control of lateral motion of the missile.

For the construction of current values of the sum (5.4.1) aboard the missile, one can use readings $V_\zeta(t)$ of the integrator of accelerations, the axis of sensitivity of which is oriented in parallel to axis ζ of the nonrotating system of coordinates $\xi\eta\zeta$ (so-called lateral integrator).

Let us note that, according to equation (1.4.1) (§ 4 of Chapter I), there occurs the relation

$$w_\zeta(t) = \frac{d^2\zeta(t)}{dt^2} = a_\zeta(t) + f_\zeta, \tag{5.4.2}$$

where $a_\zeta(t)$ - projection of apparent acceleration of the missile on axis ζ, and f_ζ - projection on the same axis of acceleration of the force of the earth's gravity. The latter, similar to equations (2.5.4), with sufficient accuracy can be represented in the form

$$f_\zeta = -\frac{f_0 R^2}{\rho^3}\zeta(t). \tag{5.4.3}$$

Thus, relation (5.4.2) can be considered the differential equation in the desired function $\zeta(t)$.

At the instant $t = 0$ the axes of the starting system of coordinates xyz, which is connected to the earth, respectively coincide with axes of the nonrotating system $\xi\eta\zeta$, and the center of masses of the missile, on assumption, is found at their common origin. Furthermore, at this instant the missile does not have velocity with respect to the starting system of coordinates xyz. Consequently, initial conditions of equation (5.4.2) are such:

$$\zeta(0) = 0, \quad \frac{d\zeta(0)}{dt} = u_\zeta(0) = u_\zeta^0, \tag{5.4.4}$$

where u_ζ^0 - velocity of the beginning of the starting system of coordinates in the nonrotating system $\xi\eta\zeta$ at the instant when

t = 0, i.e., at the beginning of the powered-flight section of its flight.

It is possible to assume that the acceleration of the force of gravity proves to have an insignificant effect on the rate of change in function $\zeta(t)$. Therefore, as a first approximation it is possible not to consider the projection f_ζ in equation (5.4.2) and, respectively, consider that

$$\frac{d^2\zeta(t)}{dt^2} = a_\zeta(t). \tag{5.4.5}$$

Hence, taking account the second initial condition (5.4.4), we have

$$\frac{d\zeta(t)}{dt} = u_\zeta(t) = V_\zeta(t) + u_\zeta^0. \tag{5.4.6}$$

Here

$$V_\zeta(t) = \int_0^t a_\zeta(t)\,dt \tag{5.4.7}$$

is the current value of the projection on axis ζ of the apparent velocity of the missile relative to the nonrotating system of coordinates $\xi\eta\zeta$. This is the reading of the integrator of accelerations with the axis of sensitivity parallel to the axis ζ. Secondary integration of both parts of equality (5.4.5), allowing also for the first of the initial conditions (5.4.4), leads to the equation

$$\zeta(t) = S_\zeta(t) + u_\zeta^0 t, \tag{5.4.8}$$

in which $S_\zeta(t)$ - projection on axis ζ of the vector of the apparent path of the missile in the nonrotating system of coordinates $\xi\eta\zeta$.

In accordance with the last equation, for the construction aboard the missile of current values $\zeta(t)$, an additional integration of readings of the aforementioned integrator of accelerations is necessary. Actually

$$S_\zeta(t) = \int_0^t V_\zeta(t)\,dt. \tag{5.4.9}$$

On the basis of equations (5.4.6) and (5.4.8) the expression for the sum (5.3.2) is reduced to the form

$$\beta_2(t) = [S_\zeta(t) + u_\zeta^0 t - \zeta^0]\frac{\partial b}{\partial \zeta} + [V_\zeta(t) + u_\zeta^n - u_\zeta^0]\frac{\partial b}{\partial u_\zeta}. \tag{5.4.10}$$

This sum, just as sum $\beta_1(t)$, obtained in § 3 of this chapter, can be constructed directly aboard the missile.

Thus, under the assumptions made about the unimportance of quantity f_ζ, it is possible by means of integrators of accelerations and computers to obtain current values of the function of lateral deviation

$$\beta(t) = \beta_1(t) + \beta_2(t). \tag{5.4.11}$$

§ 5. Calculation of the Acceleration of the Force of Gravity in the Construction of the Function of Lateral Deviation

Methods given in the fourth chapter, allow, as will be shown below, approximately integrating equation (5.4.2) for function $\zeta(t)$ taking into account the projection of the acceleration of the force of the earth's gravity f_ζ.

Let us note, first of all, that with an accuracy sufficient for practice at not too great an extension of the powered-flight section of the flight of the missile in equation (5.4.3), the distance ρ from the missile to the center of the earth can be substituted by the radius of the earth R. After this let us arrive at equation

$$f_\zeta = -\nu^2\zeta(t), \tag{5.5.1}$$

in which quantity ν (Schuler frequency) is determined by equation (4.3.4). Finally equation (5.4.2) can be replaced by the following:

$$\frac{d^2\zeta(t)}{dt^2} = a_\zeta(t) - \nu^2\zeta(t). \tag{5.5.2}$$

Let us integrate with respect to time the right and left sides of this equation within limits of 0 to t taking into account the second initial condition (5.4.4). Equating the obtained results, we arrive at integral differential relation

$$\frac{d\zeta(t)}{dt} = V_\zeta(t) - \nu^2 \int_0^t \zeta(t)\,dt + u_\zeta^0. \qquad (5.5.3)$$

Let us produce the same operation of integration again but taking into account the first initial condition (5.4.4). As a result we obtain the integral relation

$$\zeta(t) = S_\zeta(t) - \nu^2 \int_0^t \int_0^t \zeta(t)\,dt^2 + u_\zeta^0 t, \qquad (5.5.4)$$

which is the equation for the determination of lateral shift of the missile $\zeta(t)$ according to the assigned function $S_\zeta(t)$. The expression represented by equation (5.4.8) can be considered the first approximation to the solution of this equation. To obtain the following approximation, in which already the effect of acceleration of the force of gravity will be taken into account, it is possible to use the method given in the fourth chapter. First of all, let us assume in relation (5.5.4) $t = \sigma$. Then it takes the form

$$\zeta(\sigma) = S_\zeta(\sigma) - \nu^2 \int_0^\sigma \int_0^t \zeta(t)\,dt^2 + u_\zeta^0 \sigma. \qquad (5.5.5)$$

Let us further replace in the right side of equation (5.5.4) function $\zeta(t)$ by its approximation in the form of a polynomial of the second power

$$\zeta(t) = u_\zeta^0 t + \frac{\zeta(\sigma) - u_\zeta^0 \sigma}{\sigma^2} t^2, \qquad (5.5.6)$$

the coefficients of which are selected so that initial conditions (5.4.4) would be satisfied, and, furthermore, in order that the polynomial would take value $\zeta(\sigma)$ at the instant $t = \sigma$ of termination

of the powered-flight section of real motion of the missile.[1]

If then we produce a similar replacement also in relation (5.5.3) and accomplish the necessary operations of integration, then we will arrive at equality

$$\begin{aligned} u_\zeta(\sigma) &= V_\zeta(\sigma) - \nu^2 u_\zeta^0 \frac{\sigma^2}{6} - \nu^2 \zeta(\sigma) \frac{\sigma}{3} + u_\zeta^0, \\ \zeta(\sigma) &= S_\zeta(\sigma) - \nu^2 u_\zeta^0 \frac{\sigma^3}{12} - \nu^2 \zeta(\sigma) \frac{\sigma^2}{12} + u_\zeta^0 \sigma, \end{aligned} \qquad (5.5.7)$$

which can be examined as the equation for determining the value of the function itself $\zeta(\sigma)$ and its time derivative $u_\zeta(\sigma)$ at the instant $t = \sigma$.

The solution of the second equation (5.5.7) can be obtained with sufficient accuracy if in its right side quantity $\zeta(\sigma)$ is substituted by its first approximation, according to equation (5.4.8), i.e., if we assume

$$\zeta(\sigma) = S_\zeta(\sigma) + u_\zeta^0 \sigma. \qquad (5.5.8)$$

As a result we obtain equation

$$\zeta(\sigma) = \left(1 - \frac{\nu^2\sigma^2}{12}\right) S_\zeta(\sigma) + \left(1 - \frac{\nu^2\sigma^2}{6}\right) \sigma u_\zeta^0. \qquad (5.5.9)$$

We produce the same replacement in the first equation of (5.5.7). Finally we arrive at the following approximate equation for the determination of the ~~projection~~ of the actual velocity of the missile on axis ζ

$$u_\zeta(\sigma) = V_\zeta(\sigma) - \frac{\nu^2\sigma}{3} S_\zeta(\sigma) + \left(1 - \frac{\nu^2\sigma^2}{2}\right) u_\zeta^0. \qquad (5.5.10)$$

Considering the argument σ to be close to σ^*, it is possible without great loss of accuracy in equations (5.5.9) and (5.5.10)

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[1] There is a possibility, similar to § 1 of the fourth chapter, of an approximation by means of a polynomial of the third power in which the derivative when $t = \sigma$ still turns into $u_\zeta(\sigma)$.

to substitute argument σ in terms containing the small factor ν^2, by the rated value of duration of the powered-flight section σ^*. Substituting in other places of these equations argument σ for the current time t, we obtain the following final expressions for $\zeta(t)$ and $u_\zeta(t)$

$$\zeta(t)=\left[1-\frac{\nu^2(\sigma^*)^2}{12}\right]S_\zeta(t)+\left[1-\frac{\nu^2(\sigma^*)^2}{6}\right]tu^0_\zeta,$$
$$u_\zeta(t)=V_\zeta(t)-\frac{\nu^2\sigma^*}{3}S_\zeta(t)+\left[1-\frac{\nu^2(\sigma^*)^2}{2}\right]u^0_\zeta. \qquad (5.5.11)$$

Equations (5.5.11) lead to a presentation of the sum $\beta_2(t)$, more accurate in comparison with expression (5.4.10), namely:

$$\beta_2(t)=\left\{\left[1-\frac{\nu^2(\sigma^*)^2}{12}\right]S_\zeta(t)+\left[1-\frac{\nu^2(\sigma^*)^2}{6}\right]tu^0_\zeta-\zeta^*\right\}\frac{\partial b}{\partial\zeta}+$$
$$+\left\{V_\zeta(t)-\frac{\nu^2\sigma^*}{3}S_\zeta(t)+\left[1-\frac{\nu^2(\sigma^*)^2}{2}\right]u^0_\zeta-u^*_\zeta\right\}\frac{\partial b}{\partial u_\zeta}. \qquad (5.5.12)$$

The construction of function $\beta_2(t)$ aboard the missile can be produced by means of the integrator of acceleration with axis of sensitivity parallel to axis ζ (and also, of course, the integrating device).

* * *

In this book, as was already indicated in the introduction, an account of problems of the theory of inertial control of the flight of ballistic missiles was produced under the assumption of the ideal stabilization of axes of sensitivity of newtonmeters and integrators of accelerations and completely accurate operation of the latter, i.e., accurate indication by them of current values of projections of the vector of apparent acceleration of the missile and integrals of these values.

An estimate of the effect of instrumental errors of gyroscopic devices and integrators of accelerations on the magnitude of possible deviation of the missile with flight at the assigned target and also

an analysis of causes of the appearance of these errors and development of measures on their elimination represent an independent interest and fall outside the framework of this book.

Let us note, finally, that problems of inertial control of space missiles can be solved by the same methods as the problem examined above about the flight of ballistic missiles within limits of the earth.

APPENDIX

DERIVATION OF EQUATIONS FOR COSINES OF ANGLES BETWEEN AXES OF TWO SYSTEMS OF COORDINATES TURNED RELATIVE TO ONE ANOTHER AT A FINITE ANGLE

In manuals on analytical geometry and according to theoretical mechanics, equations for the cosines of angles between axes of two systems of coordinates arbitrarily turned relative to each other are derived usually with the help of several additional geometric constructions. The derivation of these equations is also possible by means of the frequent use of tables of cosines of angles between axes of the basic and auxiliary system of coordinates, each of which is turned relative to the previous around one of the coordinate axes[1] common with it. Specifically, the main axis of rotation is simultaneously the axis of coordinates of two auxiliary systems turned one relative to the other at the same angle as the main axes.

Given below is a purely analytical derivation of the mentioned equations, founded exclusively on the simplest theorems of analytical geometry, and some of these theorems are used in vectorial form.

Let us assume that the system of coordinates $\xi\eta\zeta$ is turned at an arbitrary angle ϕ around a certain axis d, which passes through

[1]See, for example, Whittaker. Analytical Kinetics. M.-L., 1937 and A. Yu. Ishlinskiy. Mechanics of Gyroscopic Systems. Izd-vo AN SSSR, 1963.

the origin of this system.[1] The final position of the system is designated by xyz. If l, m and n are cosines of angles which form the mentioned axis of the final turn d with axes of the system of coordinates $\xi\eta\zeta$, then it is obvious that the same magnitudes are respectively cosines of angles between the axis d and axes of the system xyz. Specifically, we have

$$l = \cos\alpha, \tag{1}$$

where α - angle which forms the axis of rotation d with the axis ξ of the system of coordinates $\xi\eta\zeta$ and simultaneously the same axis of rotation d with the x axis of the system xyz (see Fig. 14).

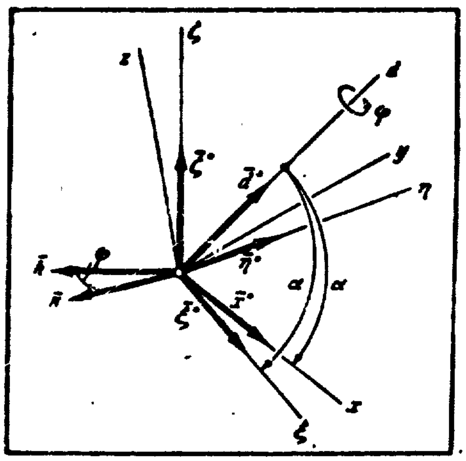

Fig. 14.

Let us designate the direction cosines of the x axis relative to the system of coordinates $\xi\eta\zeta$ respectively by a, b and c. They are in this case the desired quantities. Besides the apparent equality

$$a^2 + b^2 + c^2 = 1 \tag{2}$$

[1]In §§ 2 and 3 of the second chapter of this book such an axis was the axis of rotation of the earth, and angle φ was the product of the angular velocity of the earth U for time t, which passes from that instant when the axes of systems of coordinates $\xi\eta\zeta$ and xyz, respectively, coincided.

the direction cosines a, b and c satisfy also the relation

$$la + mb + nc = l, \tag{3}$$

which, in accordance with the known equation of the analytical geometry, expresses the magnitude of the cosine of angle α between the x axis and axis of rotation d.

For the determination of the three desired quantities a, b and c, one equation is necessary. Specifically, it can express the fact that plane xd is turned relative to plane ξd about the axis of rotation d at the assigned angle ϕ. Thus the measure of the dihedral angle between the mentioned planes also should be equivalent to angle ϕ.

Since the dihedral angle is measured by the angle between two perpendiculars to the planes forming it, then we will introduce two vectors $\bar{h}$ and $\bar{k}$, the first of which is perpendicular to the plane ξd, and the other - plane xd. According to the property of the scalar product of two vectors, we have

$$hk \cos \varphi = h_\xi k_\xi + h_\eta k_\eta + h_\zeta k_\zeta, \tag{4}$$

where h_ξ, h_η, h_ζ - projections on the axis ξ, η, ζ of vector $\bar{h}$ and, respectively, k_ξ, k_η, k_ζ - projections on these axes of vector $\bar{k}$.

Let us take as vector $\bar{h}$ the vectorial product of the single vector $\bar{\xi}^0$, located on the ξ axis, and single vector $\bar{d}^0$, which possesses the direction of the axis of rotation d. Projections of the first of them on axes ξ, η and ζ, of course, are equal, respectively, to numbers

$$1, 0, 0, \tag{5}$$

and of the second

$$l, m, n. \tag{6}$$

According to rules of the composition of the vectorial product, we obtain

$$\bar{h} = \bar{\xi}^0 \times \bar{d}^0 = \begin{vmatrix} \bar{\xi}^0 & \bar{\eta}^0 & \bar{\zeta}^0 \\ 1 & 0 & 0 \\ l & m & n \end{vmatrix}. \tag{7}$$

and, consequently,

$$h_\xi = 0, \quad h_\eta = -n, \quad h_\zeta = m. \tag{8}$$

In turn for vector $\bar{k}$ we take the vectorial product

$$\bar{k} = \bar{x}^0 \times \bar{d}^0 = \begin{vmatrix} \bar{\xi}^0 & \bar{\eta}^0 & \bar{\zeta}^0 \\ a & b & c \\ l & m & n \end{vmatrix}. \tag{9}$$

Here the second line of the determinant is made up of projections of the single vector x^0 on the axes ξ, η and ζ, equal, of course, to the direction cosines of the x axis in the system of coordinates $\xi\eta\zeta$.

Expanding the mentioned determinant, we arrive at the following equality:

$$k_\xi = nb - mc, \quad k_\eta = lc - na, \quad k_\zeta = ma - lb. \tag{10}$$

Moduli of both vectorial products (7) and (9) are identical and equal to the sine of angle α between axes ξ and d or, which is the same, between axes x and d. Thus, taking into account still another equation (1), we have

$$k = h = \sin\alpha = \sqrt{1 - l^2}. \tag{11}$$

By means of equalities (8), (10) and (11) relation (4) can be presented in the form

$$(1 - l^2)\cos\varphi = -l(mb + nc) + (n^2 + m^2)a. \tag{12}$$

It is the deficient third equation, besides equations (2) and (3), for the search of the three unknown cosines a, b and c, which determine the direction of the x axis with respect to axes of the system of coordinates $\xi\eta\zeta$.

According to equation (3), we have

$$mb + nc = l(1 - a). \tag{13}$$

Furthermore,

$$m^2 + n^2 = 1 - l^2. \tag{14}$$

Taking account the last two equalities in equation (12), we obtain after reducing similar terms to the equation for one of the desired quantities

$$a = \cos\varphi + l^2(1 - \cos\varphi). \tag{15}$$

To determine quantity b, we exclude from relation (2) cosines a and c by means of equations (12) and (15). As a result we arrive at the quadratic equation

$$\begin{gathered}[l^2 + (1 - l^2)\cos\varphi]^2 + b^2 + \frac{1}{n^2}\{l - [l^2 + \\ + (1 - l^2)\cos\varphi]\, l - bm\}^2 = 1,\end{gathered} \tag{16}$$

which after simplifications with the help of relation (14) is reduced to the form

$$b^2 - 2b\, lm\,(1 - \cos\varphi) + l^2m^2\,(1 - \cos\varphi)^2 - n^2\sin^2\varphi = 0. \tag{17}$$

Of the two roots of this equation

$$\begin{aligned} b_1 &= lm\,(1 - \cos\varphi) + n\sin\varphi, \\ b_2 &= lm\,(1 - \cos\varphi) - n\sin\varphi \end{aligned} \tag{18}$$

we should discuss the first. Actually, assuming specifically,

$$l = m = 0, \quad n = 1, \quad \varphi = \frac{\pi}{2}, \tag{19}$$

we obtain accordingly

$$b_1 = +1, \; b_2 = -1. \tag{20}$$

However, taking into account the accepted designations (see Fig. 14), we have in this case

$$b = \cos x\eta = +1 \tag{21}$$

and, consequently, the second root of equations (18) should be dropped. Thus, in general

$$b = lm\,(1 - \cos\varphi) + n\sin\varphi. \tag{22}$$

Substituting now expressions (15) and (22) for a and b into equation (3), we will arrive at the following equation for the determination of quantity c, namely:

$$c = ln\,(1 - \cos\varphi) - m\sin\varphi. \tag{23}$$

Similarly cosines of angles between the y axis and axes ξ, η, ζ and further between the z axis and the same axes ξ, η, ζ are determined. Finally, the table of cosines of the angles between axes of the system of coordinates xyz and $\xi\eta\zeta$ can be represented in the following form:

$$\begin{array}{cccc} & \xi & \eta & \zeta \\ x & (1-\cos\varphi)l^2 + \cos\varphi & (1-\cos\varphi)ml + n\sin\varphi & (1-\cos\varphi)nl - m\sin\varphi \\ y & (1-\cos\varphi)lm - n\sin\varphi & (1-\cos\varphi)m^2 + \cos\varphi & (1-\cos\varphi)mn + l\sin\varphi \\ z & (1-\cos\varphi)ln + m\sin\varphi & (1-\cos\varphi)mn - l\sin\varphi & (1-\cos\varphi)n^2 + \cos\varphi \end{array} \tag{24}$$

This table was used in the second chapter of this book.

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13. ABSTRACT

This monograph discusses the mathematical principles of certain allowed variants of inertial guidance of the flight of ballistic missiles, i.e., control without the use of any external information (radio signals, radiation of stars and others). It is assumed that the providing of the assigned flight range of the missile is produced as a result of the well-timed switching-off of its engine by a signal entering from the computer. Fed into the input of the latter are readings of sensitive elements of the system of inertial guidance, and these measure the apparent acceleration of the missile or integrals from the apparent time acceleration. Control should be such that the deviation of the actual motion of the missile from the rated does not have an effect on the range of its flight. To accomplish this measuring instruments are placed on the missile in a quite definite manner and, in particular, the direction of their axes of sensitivity in a number of cases is stabilized by means of gyroscopes. This monograph is intended for specialists in the field of the theory of control processes. It can be useful in the investigation of new problems of this discipline and also as a training manual.

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